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## Problem 6

*Let $V$ be a finite-dimensional vector space, and let $U \subset V$ be a subspace. Show that there exists a subspace $W \subset V$ such that $V = U \oplus W$.*

**Solution:** If $U = V$, then let $W = \{0\}$. In that case, it is easy to see that $U + W = V$ and $U \cap W = \{0\}$, hence $V = U \oplus W$.

Now consider the case when $U$ is strictly contained inside $V$. Extend a basis $(u_1, \ldots, u_n)$ of $U$ to a basis $(u_1, \ldots, u_n, w_1, \ldots, w_m)$ of $V$. Define $W$ to be the span of the vectors $w_1, \ldots, w_m$:

$$W \overset{\text{def}}{=} \text{Span}(w_1, \ldots, w_m).$$

It is clear that $V = U + W$. To show that $U \cap W = \{0\}$, suppose that $v \in U \cap W$. Then we can write $v = a_1 u_1 + \cdots + a_n u_n = b_1 w_1 + \cdots + b_m w_m$ for some scalars $a_i, b_i$, so that $a_1 u_1 + \cdots + a_n u_n + (-b_1) w_1 + \cdots + (-b_m) w_m = 0$. Since $(u_1, \ldots, u_n, w_1, \ldots, w_m)$ is linearly independent, this implies that each of the coefficients are zero, hence $v = 0$. We have shown that $V = U + W$ and $U \cap W = \{0\}$, which is equivalent to $V = U \oplus W$.

## Remarks

- It is not true that if $\mathcal{B}$ is a basis of $V$, then there exists a subset $\mathcal{B}' \subset \mathcal{B}$ such that $\mathcal{B}'$ is a basis of $U$. You need to start the problem by writing down a basis of $U$, and extending it to a basis of $V$.

- Some people defined $W$ to be the complement of $U$ in $V$. But the complement does not contain 0, so it would not make sense to talk about that $W$ as a vector space.

- Only a couple of people checked the case when $U = V$. The other case will generalize if you define $\text{Span}(\emptyset) = \{0\}$.