7. Let $P_n(\mathbf{R})$ be the vector space of polynomials of degree less than or equal to n, and define the following subspaces:

$$P_n^0 = \{p(x) \in P_n(\mathbf{R}) \mid p(-x) = p(x), forallx\}$$

$$P_n^1 = \{p(x) \in P_n(\mathbf{R}) \mid p(-x) = -p(x), forallx\}$$

(Note that the elements in P_n^0 are known as *even* polynomials, and the elements of P_n^1 are known as *odd* polynomials.)

Show that $P_n(\mathbf{R}) \cong P_n^0 \oplus P_n^1$. (If you uses bases for the two subspaces for this argument, then you should show that they *are* bases.)

Note that P_n^0 and P_n^1 are defined to be subspaces of $P_n(\mathbf{R})$. Thus we can prove that $P_n(\mathbf{R}) = P_n^0 \oplus P_n^1$ by showing that $P_n(\mathbf{R}) = P_n^0 + P_n^1$ and $P_n^0 \cap P_n^1 = \{0\}$. Note in this case $P_n(\mathbf{R})$ actually equals the direct sum, so the spaces are clearly isomorphic. It is also possible to prove this result by the standard method; that is, finding a bijective linear map between $P_n(\mathbf{R})$ and $P_n^0 \oplus P_n^1$ but this is not particularly difficult once you've established a basis for each space, so I will omit this proof here.

We begin by showing that $P_n^0 \cap P_n^1 = \{0\}$. Let $p \in P_n^0 \cap P_n^1$. Then by definition of these spaces $p(-x) = p(x)$ and $p(-x) = -p(x)$. Combining these statements we see that $p(x) = -p(x)$ which is only true if $p(x) = 0$.

Now we show that $P_n(\mathbf{R}) = P_n^0 + P_n^1$. The clever way is to note the following. For any $p \in P_n(\mathbf{R})$ we may write

$$p(x) = \frac{p(x) + p(-x)}{2} + \frac{p(x) - p(-x)}{2}$$

Denote $\frac{p(x)+p(-x)}{2}$ by p_0 and $\frac{p(x)-p(-x)}{2}$ by p_1. Then it is easy to check that $p_0(x) = p_0(-x)$ and $p_1(-x) = -p_1(x)$. So $p_0 \in P_n^0$ and $p_1 \in P_n^1$. Thus for every $p \in P_n(\mathbf{R})$ we have shown that $p = p_0 + p_1$ for some $p_0 \in P_n^0$ and $p_1 \in P_n^1$. Thus $P_n(\mathbf{R}) = P_n^0 + P_n^1$ and $P_n(\mathbf{R}) = P_n^0 \oplus P_n^1$.

Of course, it is also possible (and perhaps more useful) to show that $P_n(\mathbf{R}) = P_n^0 + P_n^1$ directly. We can do so by finding a basis for each space. From class we know that $\{1, x, x^2, \ldots, x^n\}$ is a basis for $P_n(\mathbf{R})$. We compute $(-x)^{2k} = x^{2k}$ for $k \in \mathbf{Z}^+ \cup \{0\}$ and see that $x^{2k} \in P_n^0$ (provided that k is small enough). Similarly $(-x)^{2k+1} = -x^{2k+1}$ so $x^{2k+1} \in P_n^1$ for small enough non-negative k. At this point it is clear that any polynomial with only even terms is even and any polynomial with only odd terms is odd, but it is not quite clear what happens in the case where p contains both even and odd terms. One way of showing this more definitively is as follows. Let $p(x) \in P_n^0$ where $p(x) = a_0 + a_1x + \ldots a_nx^n$. Then $a_0 + a_1x + \ldots a_nx^n = a_0 + a_1(-x) + \ldots a_n(-x)^n$ and with a little algebra we see that $2a_1x + 2a_3x^3 + \ldots + 2a_mx^m = 0$ where $m = 2\lceil \frac{n}{2} \rceil - 1$. As the vectors $x, x^3, \ldots, x^m$ are linearly independent, the coefficients must all be zero. Thus p contains only even terms. The proof for P_n^1 works similarly.

Now for each $p \in P_n(\mathbf{R})$ such that $p(x) = a_0 + a_1x + \ldots + a_nx^n$ it is clear that we may rearrange the terms so that $p(x) = (a_0 + a_2x^2 + \ldots + a_qx^q) + (a_1x + a_3x^2 + \ldots + a_mx^m)$ where that the first polynomial is even and the second is odd. Thus $P_n(\mathbf{R}) = P_n^0 + P_n^1$ and $P_n(\mathbf{R}) = P_n^0 \oplus P_n^1$.