Solution for HW5, part E
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Problem 7

Suppose V is a finite-dimensional vector space and U is a subspace of V. Then show $dim(V/U) = dim(V) - dim(U)$.

Solution As usual, there's more than one way to approach the problem. In what follows, let $\pi : V \rightarrow U/V$ denote the projection map, i.e. $\pi(v) = v + U$. Note that this map is linear by the definitions of addition of scalar multiplication in U/V. That is, $\pi(v) + \pi(w) = (v+U)+(w+U) = (v+w)+U = \pi(v+w)$ and $\pi(a \cdot v) = (a \cdot v)+U = a \cdot (v+U) = a \cdot \pi(v)$.

Recall that $v + U = v' + U$ iff $v - v' \in U$. Verify this if it sounds unfamiliar (hint: $v+U = \{v+u|u \in U\}$, $v'+U = \{v'+u|u \in U\}$. Why is $v-v' \in U$ a necessary and sufficient condition for the equality of the two sets?).

Proof 1: We use rank-nullity. The kernel of π is the set of elements mapping to $0+U = U$. But $v + U = 0 + U$ iff $v - 0 = v \in U$. Hence $ker(\pi) = U$. π is surjective, so $im(\pi) = V/U$.

Rank-nullity then gives us

$$dim(V) = dim(ker(\pi)) + dim(im(\pi)) = dim(U) + dim(V/U),$$

which is what we wanted to show.

Proof2:

Let $(v_1, \ldots, v_n, u_1, \ldots u_m)$ be a basis for V such that $(u_1, \ldots, u_m)$ is a basis for U. We claim $(\pi(v_1), \ldots, \pi(v_n))$ is a basis for V/U. These vectors span V/U since given $v+U \in V/U$, v may be written $v = \sum_1^m (a_i \cdot u_i) + \sum_1^n (b_j \cdot v_j)$, and $\pi(v) = v + U = \sum_1^n (b_j \cdot \pi(v_j))$.

We can check linear independence by noting that $\sum (b_j \cdot \pi(v_j)) = 0$ iff $\sum (b_j \cdot v_j) \in U$. Denoting this vector v, we have that $v \in < v_1, \ldots, v_n > \cap U$. But $V =< v_1, \ldots, v_n > \oplus U$ by our choice of basis, so $v = 0$. This implies $b_j = 0$ $(j = 1 \ldots n)$ since $v_1, \ldots v_n$ are linearly independent.

Now, $dim(V) = m + n$, $dim(U) = m$ and $dim(V/U) = n$, so we have what we wanted to show.

(note: by $< v_1, \ldots v_n >$, I mean $span(v_1, \ldots, v_n)$)