Problem Set 6, Part B – Solutions

Corina Pătraşcu

(a) $Spec(A) = Spec(B)$.

Recall that two matrices A and B are similar if there exists an invertible linear transformation S, such that $A = SBS^{-1}$ (1). Compose this relation with S^{-1} to the left and with S to the right to obtain the following relation: $S^{-1}AS = B$ (2).

Now we can go on with the proof:

- let λ be an eigenvalue for A. This means that there exists a non-zero vector v, such that $Av = \lambda v$ and applying relation (1) to v, we get that: $\lambda v = Av = SBS^{-1}v$. Now, if we compose to the left with S^{-1} it follows that $S^{-1}(\lambda v) = B(S^{-1}v)$. Using multilinearity of S^{-1} we further obtain $\lambda S^{-1}v = B(S^{-1}v)$. Now, since S is an invertible linear transformation and $v \neq 0$, we know that $S^{-1}v \neq 0$ so we can conclude that λ is an eigenvalue for B, with eigenvector $S^{-1}v$. So, $Spec(A) \subset Spec(B)$.

- now, let λ be an eigenvalue for B, with non-zero eigenvector v. Apply relation (2) to v to get that $\lambda v = Bv = S^{-1}ASv$. Now, compose to the left with S and use multilinearity of S to get that $\lambda S(v) = A(Sv)$. Again, since $v \neq 0$ and S is invertible, we know that $Sv \neq 0$, so we can conclude that λ is an eigenvalue for A, with eigenvector Sv. So, $Spec(B) \subset Spec(A)$.

In conclusion, $Spec(A) = Spec(B)$.

(b) Notation: denote the eigenspace of λ with respect to A by V_λ^A and the eigenspace of λ with respect to B by V_λ^B.

- we proved above that if v is an eigenvector of A with eigenvalue λ, then $S^{-1}v$ is an eigenvector of B with the same eigenvalue. So, $\forall v \in V_\lambda^A$ we have that $S^{-1}v \in V_\lambda^B$. We conclude that $S^{-1}(V_\lambda^A) \subset V_\lambda^B$.

- we also proved that if v is an eigenvector for B with eigenvalue λ, then Sv is an eigenvector for A with the same eigenvalue. So, $\forall v \in V_\lambda^B$ we have that $Sv \in V_\lambda^A$. We conclude that $S(V_\lambda^B) \subset V_\lambda^A$ which is equivalent to saying that $V_\lambda^B \subset S^{-1}(V_\lambda^A)$ (obtained by composing to the left with S^{-1}).

In conclusion, $S^{-1}(V_\lambda^A) = V_\lambda^B$ or, alternatively, $S(V_\lambda^B) = V_\lambda^A$.