**Math 23a Solution Set #6, Part C**

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# Problem 5

*Let $P_3$ be the vector space of polynomials of degree less than or equal to 3, with real coefficients. Let $\mathfrak{B}_1 = \{1, x, x^2, x^3\}$, $\mathfrak{B}_2 = \{1, 1+x, 1+x^2, 1+x^3\}$, and $\mathfrak{B}_3 = \{1+x, 1-x, x^2-x^3, x^2+x^3\}$ be bases for $P_3$. Let $D : P_3 \longrightarrow P_3$ be the usual differentiation operator, and let $I : P_3 \longrightarrow P_3$ be the identity.*

(a) *For each of the three bases, write down the matrix for $D$ with respect to that basis (with the basis in question considered as the basis for both the domain and range).*

**Solution:** Given a basis $\mathfrak{B} = (p_1, p_2, p_3, p_4)$ of $P_3$, the matrix of $D$ with respect to that basis is the matrix $[D]_{\mathfrak{B},\mathfrak{B}}$ whose $i$th column is $[D(p_i)]_{\mathfrak{B}}$, the coordinate vector of $D(p_i)$ with respect to $\mathfrak{B}$. Thus

$$\begin{aligned}
[D]_{\mathfrak{B}_1,\mathfrak{B}_1} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
[D]_{\mathfrak{B}_2,\mathfrak{B}_2} &= \begin{pmatrix} 0 & 1 & -2 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
[D]_{\mathfrak{B}_3,\mathfrak{B}_3} &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 1 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -1 & -1 \\ 0 & 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & -\frac{3}{2} & \frac{3}{2} \end{pmatrix}
\end{aligned}$$

(b) *Write down the matrix for $I$ where the domain has basis $\mathfrak{B}_1$ and the range has basis $\mathfrak{B}_2$.*

**Solution:**

$$[I]_{\mathfrak{B}_1,\mathfrak{B}_2} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$