Math 23a, Fall 2003

Problem Set 7, Part A
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Problem 3: *Let $V = (\mathbb{Z}/7\mathbb{Z})^3$, and consider the linear map $L : V \to V$ given by $L(x, y, z) = (x + y + z, 2y + 3z, 4z)$.*
(a) Write the matrix A for L with respect to the standard basis.
(b) Find the eigenvalues for L, and find an eigenbasis for V.
(c) Write the matrix B for L with respect to the eigenbasis.
(d) Find the change of basis matrix $S : V \to V$ that takes the standard basis to the eigenbasis.
(e) Show directly that A and B are similar.

Proof. (a) The standard basis is $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$. Since $L(e_1) = (1, 0, 0), L(e_2) = (1, 2, 4), L(e_3) = (1, 3, 4)$, we get that

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix}.$$

(b) Somehow, either randomly, or directly, or by using the procedure from the next Problem Set 8, we find that the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 4$. Then following the standard way of looking for elements in $\ker(A - \lambda I)$ for each eigenvalue we get an eigenbasis $v_1 = (1, 0, 0), v_2 = (1, 1, 0), v_3 = (5, 2, 6)$.

(c) Since $L(v_1) = (1, 0, 0), L(v_2) = (0, 2, 0), L(v_3) = (0, 0, 4)$, then the matrix B that represents L with respect to the eigenbasis is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

(d) To find the change of basis matrix from the standard basis to the eigenbasis, we need to express the elements of the standard basis in terms of the elements of the eigenbasis. We get $e_1 = v_1, e_2 = 6v_1 + v_2, e_3 = 3v_1 + 2v_2 + 6v_3$, so

$$S = \begin{pmatrix} 1 & 6 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{pmatrix}.$$

(e) Since $v_1 = e_1, v_2 = e_1 + e_2, v_3 = 5e_1 + 2e_2 + 6e_3$, we get that S^{-1}, the change of basis matrix from the eigenbasis to the standard basis, is

$$S^{-1} = \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{pmatrix}.$$

Then one directly checks that $B = S^{-1}AS$, so A and B are similar. □