Solution for HW7, part B
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Problem 4

Let $V = \mathbb{R}^n$, and let $u, v \in V$. If $A : V \to V$ is (the matrix for) a linear transformation, then define the following bilinear form:

$$f_A(u,v) = u^t A v$$

(a) Show that f_A is indeed a bilinear form.

(b) Give a necessary and sufficient condition on the matrix A that makes f_A alternating.

Solution

(a) Just check linearity in both arguments:

$$f_A(u+w,v) = (u+w)^t Av = (u^t+w^t)Av = (u^tA+w^tA)v = u^tAv+w^tAv = f_A(u,v)+f_A(w,v).$$

Also,

$$f_A(c \cdot u, v) = (c \cdot u)^t Av = c \cdot (u^t Av) = c \cdot f_A(u,v).$$

Hence, f_A is linear in its first argument. Similar reasoning shows linearity in the second argument.

(b) Claim: f_A is alternating iff $A^t = -A$.

Proof: Let a_{ij} denote the i, jth entry of the matrix A and let e_i be the ith standard basis vector for $\mathbb{R}^n$ (i.e. 0s in every entry except for the ith, which has a 1.) Suppose first that f_A is alternating. Then for all i, j, $f_A(e_i, e_j) = -f_A(e_j, e_i)$. But $f_A(e_i, e_j) = (e_i)^t Ae_j = a_{ij}$ and $f_A(e_j, e_i) = a_{ji}$ (you should check this). Hence for all i, j, $a_{ij} = -a_{ji}$, which means that $A^t = -A$.

Conversely, suppose $A^t = -A$. We saw above that this means that $f_A(e_i, e_j) = -f_A(e_j, e_i)$ for all pairs of basis vectors. It follows that $f_A(v,w) = -f_A(w,v)$ for any pair of vectors v, w, in $(R)^n$ (why? use bilinearity!). We can show this with a more explicit computation as well:

$$f_A(v,w) = v^t Aw = \sum_{i,j}(v_i a_{ij} w_j) = \sum_{i,j}(v_i(-a_{ji})w_j) = -w^t Av = -f_A(w,v).$$