Problem Set 7, Part C – Solutions

Corina Pătraşcu

The Reasoning What you have to do is find a function which is multilinear, skew-symmetric but non-alternating. Using the hint, we choose a function $f : V \to \mathbb{Z}/2\mathbb{Z}$ (please read section 30 from Halmos to understand the reasoning behind the hint). Since in $\mathbb{Z}/2\mathbb{Z}$ we have that $1 + 1 = 0$, then we can just as well construct a symmetric function because if: $f(\ldots, u, \ldots, v, \ldots) = f(\ldots, v, \ldots, u, \ldots)$ then using $1 = -1$ we get that $f(\ldots, u, \ldots, v, \ldots) = f(\ldots, v, \ldots, u, \ldots) = -f(\ldots, v, \ldots, u, \ldots)$ so it is also skew-symmetric.

The examples I give are of bilinear functions, but they can easily be generalized to multilinear functions.

Lemma 1. *The function $f : (\mathbb{Z}/2\mathbb{Z})^n \times (\mathbb{Z}/2\mathbb{Z})^n \to (\mathbb{Z}/2\mathbb{Z})$, $f(u, v) = u^t Av$ is bilinear (by the previous prob in your pset) and is symmetric if and only if $A^t = A$.*

Proof idea: If $A^t = A \Rightarrow f(u, v) = u^t Av = u^t A^t v = (v^t Au)^t = (f(v, u))^t = f(v, u)$ since $f(v, u)$ is a scalar. (Note that I used properties of the transpose: $(AB)^t = B^t A^t$). The other direction is similar. □

Now, this function satisfies the properties: multilinear, symmetric (and also skew-symmetric since we work over $\mathbb{Z}/2\mathbb{Z}$). We only need to find a matrix for which it is also non-alternating.

Particular examples

- $f : (\mathbb{Z}/2\mathbb{Z})^2 \to \mathbb{Z}/2\mathbb{Z}, f(u, v) = u \cdot v,\ \ f(1, 1) = 1 \neq 0$ – this is the trivial example.
- $f : ((\mathbb{Z}/2\mathbb{Z})^2)^2 \to \mathbb{Z}/2\mathbb{Z}, f((a, b), (c, d)) = \binom{a}{b}^t I \binom{c}{d} = ac + bd,\ \ f((1, 0), (1, 0)) = 1 \neq 0$ – note that the identity matrix satisfies the requirement of the previous proposition.
- $f : ((\mathbb{Z}/2\mathbb{Z})^2)^2 \to \mathbb{Z}/2\mathbb{Z}, f(u, v) = u^t \left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right) v,\ \ f((1, 0), (1, 0)) = 1 \neq 0.$

  Another example, not necessarily derived from the rule above, is:

- $f : ((\mathbb{Z}/2\mathbb{Z})^2)^2 \to \mathbb{Z}/2\mathbb{Z}, f((a, b), (c, d)) = ac - bd,\ \ f((1, 0), (1, 0)) = 1 \neq 0.$

Note: Please, in the future, when you make the definition of a function $f : V \to F$, make sure to say what the vector space is! From the above examples, you can see that there are *several* possibilities.