Math 23a Solution Set #7, Part D

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Problem 6

Let $\dim(V) = n$*, and let* $f : V^k \longrightarrow F$ *be an alternating* k*-linear form with* $k < n$*. Show by example that it is possible to have a set of* k *linearly independent vectors* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *in* V *such that* $f(\mathbf{v}_1, \ldots, \mathbf{v}_k) = 0$*.*

Here are two possible solutions. The first solution does not really solve the problem, since you are *given* an alternating k-linear form $f : V^k \longrightarrow F$ with $k < n$ and you're supposed to find some linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ such that $f(\mathbf{v}_1, \ldots, \mathbf{v}_k) = 0$, regardless of what the form f is. However, I didn't take off any points if you used the first solution.

Solution 1

Consider the case where $F = \mathbb{R}$, $V = \mathbb{R}^3$, and $f : (\mathbb{R}^3)^2 \longrightarrow \mathbb{R}$ is the form defined by

$$f((a_1, a_2, a_3), (b_1, b_2, b_3)) = a_1 b_2 - a_2 b_1.$$

It is easy to check that f is bilinear and alternating. You can verify this by doing the computations explicitly. Alternatively, simply observe that f is the form $f_A : (\mathbb{R}^3)^2 \longrightarrow \mathbb{R}$, where

$$f_A(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t A \mathbf{v}, \qquad \text{where } A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which you know is bilinear and alternating from Problem 4. Now, we know that the vectors $(1, 0, 0)$ and $(0, 0, 1)$ are linearly independent, but

$$f((1, 0, 0), (0, 0, 1)) = 1 \cdot 0 - 0 \cdot 0 = 0.$$

Solution 2

(*Due to Tien Anh Nguyen.*) Suppose V has basis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ where $n \geq 3$, and let $f : V^k \longrightarrow F$ be any alternating k-linear form with $k < n$. If $f(\mathbf{v}_1, \ldots, \mathbf{v}_k) = 0$, then we are done. Otherwise, $f(\mathbf{v}_1, \ldots, \mathbf{v}_k) = a$ for some $a \in F$ where $a \neq 0$. Similarly, if

$f(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}) = 0$, we are done, so suppose that $f(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}) = b$ where $b \neq 0$. Since F is a field, there exists α such that $a = \alpha b$. Using multi-linearity, we get

$$\begin{aligned} f(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_k - \alpha \mathbf{v}_{k+1}) &= f(\mathbf{v}_1, \ldots, \mathbf{v}_k) - \alpha f(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}) \\ &= a - \alpha b \\ &= 0. \end{aligned}$$

Now we are done, since the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_k - \alpha \mathbf{v}_{k+1}$ are linearly independent.

Remarks

- Many people had a solution like "if $f : (\mathbb{R}^3)^2 \longrightarrow \mathbb{R}$ is the form that measures 2-volume in the xy-plane, then $f(e_1, e_3) = 0$". This is fine, but I wanted you to write out an explicit expression for the form, then prove it is bilinear and alternating. Otherwise, it is not clear to me that you understand what that form is.

- Over the course of the last few problem sets, I have noticed that many people simply copy their solutions from other people almost word for word. This is particularly obvious since the people who copy their solutions from other people normally hand their problem set in immediately after the other person, so I grade several consecutive problem sets with almost exactly the same solution. This is not good. If I catch people doing this on a future problem set you will get a score of 0.