

7. Let V be a vector space over F . Consider the set of k -linear forms $f : V^k \rightarrow F$. For any two such forms f_1 and f_2 and any scalar $c \in F$, we define:

$$(f_1 + f_2)(v_1, \dots, v_k) = f_1(v_1, \dots, v_k) + f_2(v_1, \dots, v_k), \forall v_1, \dots, v_k \in V$$

$$(cf_1)(v_1, \dots, v_k) = c \cdot f_1(v_1, \dots, v_k), \forall v_1, \dots, v_k \in V$$

b. Let $V = \mathbf{R}^2$. Show that the form $f : V^2 \rightarrow \mathbf{R}$ defined by $f((a, b), (c, d)) = ad - bc$ is bilinear and alternating.

First we check that f is alternating. For any $(a, b) \in \mathbf{R}^2$, $f((a, b), (a, b)) = ab - ba = ab - ab = 0$. Thus we conclude that f is alternating (and skew-symmetric from a result in class).

Now we show that f is bilinear. First we check the additive condition for any $(a, b), (c, d), (e, f) \in \mathbf{R}^2$. We see that $f((a, b) + (c, d), (e, f)) = f((a + c, b + d), (e, f)) = (a + c)f - (b + d)e = (af - be) + (cf - de) = f((a, b), (e, f)) + f((c, d), (e, f))$. To show additivity in the second position, we note that $f((e, f), (a, b) + (c, d)) = -f((a, b) + (c, d), (e, f)) = f((a, b), (e, f)) - f((c, d), (e, f)) = f((e, f), (a, b)) + f((e, f), (c, d))$.

Finally we check the multiplicative condition for any $(a, b), (c, d) \in \mathbf{R}^2$ and $\alpha, \beta \in \mathbf{R}$. We see that $f(\alpha(a, b), \beta(c, d)) = f((\alpha a, \alpha b), (\beta c, \beta d)) = (\alpha a)(\beta d) - (\alpha b)(\beta c) = \alpha\beta(ad - bc) = \alpha\beta f((a, b), (c, d))$ as desired. So f is indeed an alternating bilinear form.

c. Now let $V = \mathbf{R}^3$. Construct two *linearly independent* alternating bilinear forms $f : V^2 \rightarrow \mathbf{R}$.

Any two of the basis vectors for the space of alternating bilinear forms on $V = \mathbf{R}^3$ constructed in part d satisfy this requirement. (Yes you can do this, provided that part d does not depend on part c.)

d. Determine the dimensions of the spaces of alternating bilinear forms on $V = \mathbf{R}^2$ and $V = \mathbf{R}^3$.

From problem 4 on this problem set we may write any alternating bilinear form on an n -dimensional vector space as $f(u, v) = u^t A v$ for some matrix $A \in M_n(F)$ such that $A^t = -A$. When $V = \mathbf{R}^2$ it is easy to see that such a matrix must look like

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ for some } a \in \mathbf{R}$$

Because each alternating bilinear form on \mathbf{R}^2 is characterized by a 2×2 real matrix and each such matrix is a scalar multiple of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ it is easy to see that its corresponding form f_0 serves as a basis for this space and consequently the dimension of the space of

alternating bilinear forms on \mathbf{R}^2 is one-dimensional. Explicitly, for any f in this space and its corresponding matrix $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$

$$f = u^t A v = u^t \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} v = u^t a \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} v = a f_0$$

Our computation for $V = \mathbf{R}^3$ is similar. It is easy to see that the requirement $A^t = -A$ for $A \in M_3(\mathbf{R})$ means that the matrices of alternating bilinear forms on \mathbf{R}^3 must take the form

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = a \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \text{ for some } a, b, c \in \mathbf{R}$$

Clearly these matrices are linearly independent and thus their corresponding forms are as well. Additionally it is clear that any alternating bilinear form f on \mathbf{R}^3 may be written as a linear combination of the forms f_0 , f_1 , and f_2 corresponding to these matrices, though this fact can be verified as we did above. This verification can also be done strictly algebraically, though it's a bit of a mess. Here the matrices make computations easier, in addition to illustrating why the dimension is $\binom{n}{2} = \frac{n(n-1)}{2}$ as many of you noted. Namely, there are $\frac{n(n-1)}{2}$ entries above the diagonal in an $n \times n$ matrix, which each give rise to a different basis element.

So f_0 , f_1 , and f_2 form a basis for this space and the dimension is 3.