2. Recall the definition of the *transpose* of a matrix, as referred to in homework problem #
8.8, and prove the following:

Theorem 1. *If A is an $n \times n$ matrix, then $\det A^t = \det A$.*

As many of you noticed, there is a proof of this in your text, albeit a computationally
ugly one. However, there is a much simpler proof by induction and almost everyone who
figured this received full credit for this problem.
We induct on the dimension of the matrix and start with a trivial base case. When $n = 1$,
$A = A^t$ so of course $\det A = \det A^t$.
Now we assume this is true when $n = k$ and show that it must be true for $n = k+1$. Let
A be a $k+1 \times k+1$ matrix indexed by (a_{ij}). Then we compute the determinant of A by ex-
panding by minors along the first row. Thus $\det A = a_{11} \det A_{11} + \ldots + a_{1n} \det A_{1n}$. Similarly,
we expand A^t by minors along the first *column* to get $\det A^t = a^t_{11} \det A^t_{11} + \ldots + a^t_{n1} \det A^t_{n1}$.
But we know that $a^t j1 = a_{1j}$ and it is easy to check that $A^t_{j1} = A_{1j}$ from the explicit formu-
las for minors. Because these matrices are $k \times k$ our inductive hypothesis shows that their
determinants are equal. So $\det A = \det A^t$ and we are done.

4. Show that $A : V \to V$ is invertible if and only if $\det A \neq 0$. (We have used this fact
several times already, including in problem # 3. The point of this exercise is to make you
think carefully about the steps we used when we made the transition from alternating forms
to determinants.)

The biggest danger in this problem is stated in the hint. Many of the results we have
proven so far about matrices and determinants (including problem 3 and the multiplicity of
determinants!) already used this result. So you can't use them to prove it (otherwise that
would be circular reasoning). So there really is no other way than to resort back to the
definition of determinants with alternating forms. So let d be the alternating $n-$ linear form
such that $d(e_1, \ldots, e_n) = 1$. Then recall that $\det A$ is defined to be $d(Ae_1, \ldots, Ae_n)$ where
Ae_k is the kth column of A.
($\Rightarrow$) If A is invertible then the columns of A must be linearly dependent. For if there
existed scalars, not all zero, such that $c_1 Ae_1 + \ldots + c_n Ae_n = 0$ then if we let v be the
column vector $(c_1, \ldots, c_n)^t$, $Av = 0$ and $v \neq 0$. So A would not be injective, a contradiction
because only bijective maps are invertible. So we can conclude that $Ae_1, \ldots, Ae_n$ are linearly
independent. But then from a fact proven in class $d(Ae_1, \ldots, Ae_n) \neq 0$.
($\Leftarrow$) Conversely, if $\det A \neq 0$, then $d(Ae_1, \ldots, Ae_n) \neq 0$ and another fact from class
shows that $Ae_1, \ldots, Ae_n$ are linearly independent (because if they were linearly dependent,
the alternating form would vanish). So A must be injective and as $\dim V = n$ these vectors
also span. So A is also surjective and thus invertible.