Math 23a, Fall 2003

Problem Set 8, Part B
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Problem 3: *Let $A : V \to V$ be a linear transformation on a finite dimensional vector space, and by slight abuse of notation, let A also be the matrix for this transformation with respect to a fixed basis. Using the following method we determine the eigenvalues of A: λ is an eigenvalue for $A \Leftrightarrow V_\lambda = \ker(A-\lambda I)$ is non-trivial $\Leftrightarrow A-\lambda I$ is not invertible $\Leftrightarrow det(A-\lambda I) = 0$. Thus we are inspired to make the following definition: $p_A(\lambda) = det(A-\lambda I)$ is called the characteristic polynomial of A. The eigenvalues of A will be the roots of the characteristic polynomial.*
(a) Prove that no scalar $\lambda_0 \in F$ is an eigenvalue for A unless it is a root of $p_A(\lambda)$.
(b) If $p_A(\lambda) = (\lambda-\lambda_0)^k q_A(\lambda)$ with $q(\lambda_0) \neq 0$, then we say that the eigenvalue λ_0 has algebraic multiplicity equal to k. Show that the geometric multiplicity, which by definition is the dimension of the corresponding eigenspace, of an eigenvalue is less than or equal to the algebraic multiplicity.
(c) Use this method to find the eigenvalues of the real matrix A

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Is the matrix A diagonalizable? Explain.

Proof. (a) As we showed in the statement of the problem λ_0 is an eigenvalue $\Leftrightarrow p_A(\lambda_0) = 0$. (a) is equivalent to this being the contrapositive of this statement.

(b) Let λ_0 be an eigenvalue of A with geometric multiplicity k. This, as we know, means that there is a basis of the eigenspace V_{λ_0} of the form $\{v_1, \dots, v_k\}$. Then, using another standard fact from class, we can extend this basis to a basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of V.

Then the matrix representation of A is $A = [Av_1, Av_2, \ldots, Av_n]$, i.e.

$$\begin{pmatrix} \lambda_0 & 0 & \ldots & 0 & a_{1,k+1} & \ldots \\ 0 & \lambda_0 & \ldots & 0 & a_{2,k+1} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & a_{3,k+1} & \ldots \\ 0 & 0 & \ldots & \lambda_0 & a_{n,k+1} & \ldots \end{pmatrix},$$

since $Av_i = \lambda_0 v_i$, for $i = 1, \ldots, k$.

Then

$$A - \lambda I = \begin{pmatrix} \lambda_0 - \lambda & 0 & \ldots & 0 & a_{1,k+1} & \ldots \\ 0 & \lambda_0 - \lambda & \ldots & 0 & a_{2,k+1} & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & a_{3,k+1} & \ldots \\ 0 & 0 & \ldots & \lambda_0 & a_{n,k+1} & \ldots \end{pmatrix}.$$

We can now compute the determinant $p_A(\lambda)$ of $A - \lambda I$. We use cofactor expansion along the first column, to get $p_A(\lambda) = (\lambda_0 - \lambda)\det A_1 1$, where $A_1 1$ is the corresponding minor (the lower right $(n-1) \times (n-1)$ square). Then we apply the same procedure again and again, a total of k times, to get that $p_A(\lambda) = (\lambda_0 - \lambda)^k \det B$, where B is the remaining $(n-k) \times (n-k)$ matrix in the lower right corner of A. Thus λ_0 is a root of multiplicity at least k, which proves that the geometric multiplicity is never greater than the algebraic multiplicity.

(c) We can directly compute the characteristic polynomial, which turns out to be $p_A(\lambda) = (\lambda - 1)(\lambda + 1)(\lambda^2 + 1)$. Since A is a real matrix, we are only interested in the real eigenvalues $+1$ and -1. As we can see their algebraic multiplicities are both one, so the sum of their geometric multiplicities cannot be more than two, by part (b), so A is not diagonalizable. By the way, this matrix is a good example of a matrix, which is not diagonalizable when considered as a real matrix, but is diagonalizable when considered as a complex matrix. That is why you should always be careful what your base field is. □