Problem Set 8, Part D – Solutions

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8. Recall the definition of multiplication of Cauchy sequences given in class: if $\{a_n\}_n$ and $\{b_n\}_n$ are two Cauchy sequences, then: $[\{a_n\}][\{b_n\}] = [\{a_n \cdot b_n\}]$.

Then, the multiplicative identity suggests itself as: $I = [\{1\}_n]$. The infinite sequence of 1's is clearly Cauchy since the difference of each two terms is 0, and therefore smaller then any positive epsioln for all $n \geq 1$. Moreover, it converges to 1.

Also, given a real number x represented by the equivalence class: $[\{a_n\}]$ we have that $x \cdot I = [\{a_n\}] \cdot [\{1\}] = [\{a_n \cdot 1\}] = [\{a_n\}] = x$. In conclusion, I is the multiplicative identity. □

9. Given a non-zero, real number x, say it is defined by the equivalence class $[\{a_n\}]$. We want to find its multiplicative inverse $x^{-1} = [\{b_n\}]$.

Note that even though $x \neq 0$, this doesn't imply that $\{a_n\}$ contains only non-zero terms. In fact, it can contain zeroes, but only finitely many of them. And this is because: $\forall \epsilon > 0, \exists N$ such that $|a_n - x| < \epsilon$, $\forall n > N$. Taking $\epsilon = |x|/2$ we conclude that there exists N such that $\forall n > N, |a_n - x| < |x|/2$ which implies $-|x|/2 < a_n - x < |x|/2$. So, $\forall n > N, |a_n| > |x|/2 > 0$. In conclusion, from a point on, a_n will contain only non-zero elements.

Let $b_n = 0, \forall n \leq N$ and $b_n = 1/a_n, \forall n > N$. This is well-defined.

Moreover, $[\{a_n\}] \cdot [\{b_n\}] = [\{0, ..., 0, 1, ...\}]$ which is clearly the identity since the representative sequence contains finitely many zeroes and infinitely many ones.

The only thing left to prove is that b_n is a Cauchy sequence and for that we have to show that given any $\epsilon > 0$, there exists N_1 such that $|b_m - b_n| < \epsilon$, $\forall m, n > \epsilon$. But this is equivalent to finding N_1 such that $|1/a_m - 1/a_n| < \epsilon$, $\forall m, n > N_1$. Note that the first N terms are not interesting to us since they are all 0.

We know that $\{a_n\}$ is Cauchy, so, given $\epsilon' = \epsilon \cdot |x|^2/4$ there exists N_2 such that $|a_n - a_m| < \epsilon \cdot |x|^2/4$ for all $m, n > N_2$.

Moreover, we know from above, that $\forall n > N$ we have $|a_n| > |x|/2$.

So, for all $n > max(N, N_1)$ we have that $|1/a_n - 1/a_m| = \frac{|a_n - a_m|}{|a_n a_m|} < \epsilon \cdot \frac{|x|^2}{4} \cdot \frac{1}{|x|^2/4} < \epsilon$ which proves that $\{b_n\}$ is a Cauchy sequence and is the multiplicative inverse of the real, non-zero number x. □