Math 23a Solution Set #8, Part E

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Problem 10

Use the definition of Cauchy sequence to show that the sequence of rational numbers $\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$ *is a Cauchy sequence.*

Solution: Let $\epsilon > 0$. We want to exhibit an $N \in \mathbb{N}$ such that $|\frac{1}{n^2} - \frac{1}{m^2}| < \epsilon$ whenever $n, m \geq N$. Take $N = \left\lceil \sqrt{1/\epsilon} \right\rceil$ (here $\lceil x \rceil$ denotes the *ceiling* of $x \in \mathbb{R}$, i.e. the smallest integer greater than or equal to x). Then, for $n, m \geq N$, we get

$$\left|\frac{1}{n^2} - \frac{1}{m^2}\right| < \max\left\{\frac{1}{n^2}, \frac{1}{m^2}\right\} \leq \frac{1}{\left\lceil \sqrt{1/\epsilon} \right\rceil^2} \leq \frac{1}{(\sqrt{1/\epsilon})^2} = \epsilon,$$

which proves the claim.