Math 23a Solution Set #9, Part A

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Problem 6

In this problem, we will show that the golden ratio, $\varphi = \frac{1+\sqrt{5}}{2}$, *is a real number because it is the supremum of a non-empty bounded set of real numbers. (More precisely, we will show that φ is the limit of a bounded, increasing sequence.)*

Consider the recursively defined sequence:

$$a_1 = 1 \quad \textit{and} \quad a_{n+1} = \sqrt{1 + a_n}, \ \textit{for } n \geq 1.$$

(a) *Use induction to show that $a_n \leq 2$ for all $n \in \mathbb{N}$*

Solution: For the base case, simply note that $a_1 = 1 \leq 2$. For the inductive hypothesis, assume that $a_n \leq 2$ for some $n \in \mathbb{N}$. Then

$$a_{n+1} = \sqrt{1 + a_n} \leq \sqrt{1+2} = \sqrt{3} \leq \sqrt{4} = 2.$$

(b) *Use induction to show that $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$*

Solution: Certainly $a_1 = 1$ is less than or equal to $a_2 = \sqrt{2}$. Assume that $a_n \leq a_{n+1}$ for some $n \in \mathbb{N}$. Then

$$a_{n+1} = \sqrt{1 + a_n} \leq \sqrt{1 + a_{n+1}} = a_{n+2}.$$

(c) *Since $L = \lim_{n\to\infty} a_n$ exists by the completeness axiom for $\mathbb{R}$, show that L satisfies the equation $L^2 - L - 1 = 0$ by considering the expression $\lim_{n\to\infty} a_{n+1}^2 - a_n - 1$.*

Solution: Recall that for any sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$, we have $\lim_{n\to\infty}(a_n + b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$ and $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$ provided that these limits exist. So, given that $L = \lim_{n\to\infty} a_n$ exists, we have

$$\begin{aligned}
L^2 - L - 1 &= \left(\lim_{n\to\infty} a_n\right)^2 - \left(\lim_{n\to\infty} a_n\right) - 1 \\
&= \left(\lim_{n\to\infty} a_{n+1}\right)^2 - \left(\lim_{n\to\infty} a_n\right) - 1 \\
&= \left(\lim_{n\to\infty} a_{n+1}^2\right) - \left(\lim_{n\to\infty} a_n\right) - 1
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} (a_{n+1}^2 - a_n - 1) \\
&= \lim_{n\to\infty} ((\sqrt{1+a_n})^2 - a_n - 1) \\
&= \lim_{n\to\infty} 0 \\
&= 0.
\end{aligned}
$$

(d) *Show that* $L = \varphi$.

Solution: Since L satisfies the equation $L^2 - L - 1$, we know that $L = \varphi$ or $L = \frac{1-\sqrt{5}}{2}$. Since L is the supremum of a nonempty set of positive real numbers, the second solution is impossible, so we conclude that $L = \varphi$.

Remarks

- Many people attempted (and failed!) to justify that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$. This was unnecessary, but here is how you would do this using the limit definition. Suppose that $\lim_{n\to\infty} a_n = L$. Then for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|L - a_n| < \epsilon$ whenever $n \geq N$. But then $|L - a_{n+1}| < \epsilon$ whenever $n \geq N$ as well, which is exactly the statement that $\lim_{n\to\infty} a_{n+1} = L$.