

6. Recall homework problem #7.4 in which you showed that with $V = \mathbf{R}^2$ and $u, v \in V$, if $A : V \rightarrow V$ is a linear transformation, then

$$f_A(u, v) = u^t Av$$

defines a bilinear form.

Give a necessary and sufficient condition on A that makes f_A an inner product. (Full points for a complete answer in the $n = 2$ case.)

We will discover the conditions, while proving their necessity. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the matrix of an inner product and let $u = (u_1, u_2)^t, v = (v_1, v_2)^t \in V$. From the symmetry of a inner product, we require that $u^t Av = v^t Au$ so $au_1v_1 + bu_1v_2 + cu_2v_1 + du_2v_2 = au_1v_1 + bu_2v_1 + cu_1v_2 + du_2v_2$, i.e., that $bu_1v_2 + cu_2v_1 = bu_2v_1 + cu_1v_2$. This is true for all u and v iff $b = c$ so this is our first condition.

Next we compute some specific cases. An inner product must be positive definite so $e_1^t Ae_1 = a$ must be greater than zero because $e_1 \neq 0$. So we require $a > 0$. Similarly $e_2^t Ae_2 = d > 0$.

Finally, we check the tricky case of $v^t Av = ax^2 + 2bxy + dy^2$ where $v = (x, y)^t \neq 0$. We know that $ax^2 + 2bxy + dy^2$ must be positive. We now fix y and consider the quadratic polynomial $ax^2 + 2bxy + dy^2$ as a function of x . From basic algebra we know that this polynomial has a root unless the discriminant is negative. So we must have $(2b)^2 y^2 - 4acy^2 < 0$ so $b^2 < ac$. Otherwise for nonzero y this polynomial would have a non-trivial solution and $f(v, v) = 0$ for some $v \neq 0$. When we fix x and consider $ax^2 + 2bxy + dy^2$ as a function of y we get the same result so $b^2 < ac$ is our final condition.

We now complete the proof by showing that these conditions on $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ are sufficient. So now let A be a symmetric matrix such that $a, d > 0$ and $b^2 < ac$ (i.e., $\det(A) > 0$). We must show that A is the matrix of an inner product. From problem set 7 we know that a bilinear form $f(u, v) = u^t Av$ so A is the matrix of a bilinear form. Thus we need only check that the form defined in this way is symmetric and positive-definite. The symmetric condition is easy and it follows directly from our demonstration of the necessity that $b = c$ above.

It remains to show that f is positive definite. Let $v = (x, y)^t \in V$ such that $v \neq 0$. Then $f(v, v) = v^t Av = ax^2 + 2bxy + dy^2$. We let y take the value given by v and let x vary. Thus we have a quadratic in x with a negative discriminant. Thus the quadratic has no roots and in particular $f(v, v) \neq 0$. Because polynomials are continuous and this quadratic does not intersect the x axis, $ax^2 + 2bxy + dy^2$ must either be entirely positive or entirely negative. To check this we consider the point when $x = 0$. Then the quadratic becomes dy^2 and $d > 0$ so this is clearly positive. (Note there is a slight ambiguity in the case where $y = 0$ but a similar analysis works for fixed x and at least one of these must be nonzero since $v \neq 0$). So f is positive definite and our proof is complete.