Math 23a, Fall 2003

Problem Set 9, Part C
Solutions written by Tseno Tselkov

Problem 8: *Consider the real vector space* $V = C[0,1]$ *of continuous real-valued functions defined on the closed interval* $[0,1]$*, and define the bilinear form* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ *by:*

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

In class, we checked that this form is indeed bilinear, and it is clear that it is symmetric and positive. Use facts from single-variable Calculus to prove that this form is positive-definite.

Proof. To show that the given bilinear form is positive-definite, we clearly only need to show that if $f(x)$ is not identically zero, then

$$\langle f, f \rangle = \int_0^1 f^2(x)dx > 0.$$

Since $f(x)$ is not identically zero, there is $a \in [0,1]$ such that $f(a) = \varepsilon \neq 0$. Now since $f(x)$ is continous, by the definition of continuity, we get that there exists $\delta > 0$, such that

$$|f(x) - f(a)| < |\varepsilon|\,,\ \forall x \in [a-\delta, a+\delta].$$

Then since $f^2(x) \geq 0$ we get

$$\int_0^1 f^2(x)dx \geq \int_{a-\delta}^{a+\delta} f^2(x)dx > 0.$$

The last inequality follows from the fact that clearly $f^2(x) > 0$ for all $x \in [a-\delta, a+\delta]$, as we guaranteed above that $f(x) \neq 0$ for all $x \in [a-\delta, a+\delta]$. Thus, the given bilinear form is indeed positive-definite. □