Solution for HW9, part D
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Problem 4

Let V be a vector space with inner product $< \cdot, \cdot >$ and associated norm $\| \cdot \|$. Recall that a linear transformation $A : V \to V$ is *norm-preserving* if $\| Av \| = \| v \|$ for every $v \in V$ and *inner-product-preserving* if $< Au, Av > = < u, v >$ for all $u, v \in V$.

(a) Show A is inner-product-preserving iff it is norm-preserving.

(b) Let $V = \mathbb{R}^2$ with the usual inner product. Find all norm-preserving linear transformations/matrices.

Solution

(a) Suppose A is norm-preserving. Then for all $v, w \in V$, we have

$$\| Av + Aw \|^2 = < Av + Aw, Av + Aw > = \| v + w \|^2 = < v + w, v + w > .$$

Expanding both sides of the equation using bilinearity, we get:

$$\| Av \|^2 + 2 < Av, Aw > + \| Aw \|^2 = \| v \|^2 + 2 < v, w > + \| w \|^2,$$

which implies $< Av, Aw > = < v, w >$.

Conversely, if A is inner-product-preserving, then for all $v \in V$,

$$< Av, Av > = \| Av \|^2 = < v, v > = \| v \|^2 .$$

(b) I'll find all norm-preserving matrices. A matrix $A : \mathbb{R}^2 \to \mathbb{R}^2$ is norm-preserving iff it is inner-product-preserving by part (a). Let (e_1, e_2) be the standard basis for $\mathbb{R}^2$. Note that a map is inner-product-preserving iff it is inner-product-preserving on all pairs of basis vectors, i.e. iff $< Ae_i, Ae_j > = < e_i, e_j >$ for all i, j (this statement holds for any basis set, not just the standard basis–check it!). Now (e_1, e_2) is an orthonormal family, that is $< e_1, e_2 > = 0$ and $< e_i, e_i > = 1 (i = 1, 2)$. Since Ae_i is the ith column of A, we see that A is inner-product-preserving iff the column vectors of A form an orthonormal family.

Let a_1 be the first column vector of A and a_2 the second. a_1 is normal iff it lies on the unit circle, i.e. iff

$$a_1 = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

for some $\theta \in [0, 2\pi)$. Now a_2 must be normal and orthoganal to a_1 and there are precisely two vectors satisfying these conditions: a_1 rotated $\pi/2$ in the clockwise or the counter-clockwise direction. That is,

$$a_2 = \pm \begin{pmatrix} -sin(\theta) \\ cos(\theta) \end{pmatrix}.$$

Thus we have two distinct collections of matrices which together form the set of all norm-preserving matrices:

$$\left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

and

$$\left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} : \theta \in [0, 2\pi) \right\}.$$

In the first collection are the rotations in $\mathbb{R}^2$ and are *orientation-preserving* ($det = 1 > 0$). The second collection consists of *orientation-reversing* maps ($det = -1 < 0$) and every element can be written as a rotation composed with a reflection.