Problem Set 9, Part E – Solutions

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a. In order to prove that the given map is a norm, we have to check the three properties given in the statement of the problem.

i. $\|c \cdot v\| = \max\{|cx_1|, ..., |cx_n|\} = \max\{|c| \cdot |x_1|, ..., |c| \cdot |x_1|\} = |c| \max\{|x_1|, ..., |x_n|\} = |c| \cdot \|v\|$.

ii. $\|v\| \geq 0$, clearly, since its the maximum of some absolute values which are all positive. Now, if $\|v\| = 0$, then all the absolute values have 0 both as lower as well as upper bound $\Rightarrow x_i = 0, \forall i \Rightarrow v = 0$. If $v = 0$, it is clear that $x_i = 0,\ \forall i \Rightarrow \|v\| = 0$.

iii. let $u = (x_1, ..., x_n)$ and $v = (y_1, ..., y_n)$. Then $\|u + v\| = \max\{|x_1 + y_1|, ..., |x_n + y_n|\}$. Wlog, say $\|u + v\| = |x_i + y_i|$. By the triangle inequality for real numbers, we know that $|x_i + y_i| \leq |x_i| + |y_i|$ which implies that $\|u + v\| \leq |x_i| + |y_i| \leq \|u\| + \|v\|$. □

b. Assume that there is an inner product associated with the given norm $\Rightarrow \|v\| = \sqrt{< v, v >}$. Then, by bilinearity, we would have:

$\|u + v\|^2 =< u + v, u + v >=< u, u + v > + < v, u + v >=< u, u > + < u, v > + < v, u > + < v, v >= \|u\|^2 + \|v\|^2 + 2 < u, v >$, by symmetry. Similarly, $\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2 < u, v >$. Adding these two equations, we obtain:

$\|u + v\|^2 + \|u - v\|^2 = \|u\|^2 + \|v\|^2$. And this relation has to hold for all $u, v \in V$.

But taking $u = (1, 0, ..., 0)$ and $v = (0, 1, 0, ..., 0)$ (which exist since we are in the case $n > 1$) and plugging them in the previous equation, we obtain: $1 + 1 = 2 + 2 \Rightarrow 2 = 4$ which is a contradiction.

So, it is imposible to associate an inner product with the given norm. □