

Last Name: _____

First Name: _____

MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Final Exam (in-class portion)
January 19, 2001

Directions: You have three hours for this exam, though I have designed it to take less than the full amount of time. No calculators, notes, books, etc. are allowed. Please answer on the pages provided—there is a blank page at the end for scratch work. Note that not all questions are equally weighted.

Problem	Points	Score
1	36	
2	24	
3	15	
4	10	
5	15	
6	10	
7	10	
8	0*	
Total	120	

* Problem #8 is just something to think about. It is **not** extra credit.

1. True or False (36 points, 2 each)

- T** or **F** Every field has either a prime number of elements or infinitely many elements.
- T** or **F** The columns of an $n \times n$ matrix A are linearly independent if and only if A is invertible.
- T** or **F** If A and B are linear maps from a vector space V to itself and $AB = I$, then $B = A^{-1}$.
- T** or **F** If U and V are vector spaces over the same field F , then $(U \oplus V)/V \cong U$.
- T** or **F** If $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation and $n > m$, then $\text{Ker}(L)$ is non-trivial.
- T** or **F** If V is a vector space over the field $\mathbb{Z}/p\mathbb{Z}$ for some prime p , then V is finite-dimensional.
- T** or **F** If $A : U \rightarrow V$ and $B : V \rightarrow W$ are both surjective linear maps, then $B \circ A : U \rightarrow W$ is also surjective.
- T** or **F** If V is a vector space over \mathbb{R} and $\dim(V) = 3$, then the space of alternating 2-linear forms on V is one-dimensional.
- T** or **F** If $A, B \in M_n(\mathbb{R})$, then $\det(A+B) = \det(A) + \det(B)$.
- T** or **F** The union of any collection of closed sets is closed.
- T** or **F** If $\{C_n\}$ is a nested sequence of compact sets in \mathbb{R}^n , then $\bigcap C_n$ is compact and non-empty.
- T** or **F** Every metric space has an associated norm.
- T** or **F** If V is a subspace of an inner product space U over \mathbb{R} , then $V \cap V^\perp = \{\mathbf{0}\}$.
- T** or **F** If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous and $S \subset \mathbb{R}^m$ is disconnected, then $f^{-1}(S)$ is disconnected in \mathbb{R}^n .
- T** or **F** If $A \subset \mathbb{R}^n$ is open and $\mathbf{x} \in A$, then there is some $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \subset A$.
- T** or **F** If $A \subset \mathbb{R}^n$ is a closed set and $\mathbf{x} \in A$, then for every $\varepsilon > 0$, the ball $B_\varepsilon(\mathbf{x})$ contains points outside of A .
- T** or **F** If V is an inner-product space (over \mathbb{R}) and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for V , then $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ when $i \neq j$.
- T** or **F** The sequence $\{a_n\}$ given by $a_n = 6^n$ is a Cauchy sequence in \mathbb{Q}_3 .

2. True or False with Justification (24 points, 4 each)

Decide whether the following statements are true or false and give a *brief* justification—write no more than one sentence! (For example, if a statement is true, you might cite a theorem or provide a one-line proof. If a statement is false, you might provide a counter-example.)

T or **F** \mathbb{Z} is bounded as a subset of \mathbb{Q}_p .

T or **F** If $A : U \longrightarrow V$ is a linear transformation and \mathbf{u} and \mathbf{v} are eigenvectors of A with distinct eigenvalues, then \mathbf{u} and \mathbf{v} are linearly independent.

T or **F** If V is a finite-dimensional vector space spanned by a set of n vectors, then $\dim(V) = n$.

T or **F** If V is an inner-product space with basis \mathfrak{B} , then it is possible to find an orthonormal basis for V .

T or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous and $S \subset \mathbb{R}^n$ is open, then $f(S)$ is open in \mathbb{R}^m .

T or **F** Every bounded infinite subset of \mathbb{R}^n has an accumulation point.

3. Examples (15 points, 5 each)

Provide examples of the following. Justification is not required as long as the examples are correct.

- (a) A subset $S \subset \mathbb{R}^n$ that is neither open nor closed.
- (b) A vector space V and a linear transformation $A : V \longrightarrow V$ that has infinitely many eigenvalues.
- (c) A metric space S and a non-trivial subset A that is both open and closed.

4. Similar Matrices (10 points)

- (a) Let $A, B \in M_n(F)$. Define what it means for A and B to be *similar*.
- (b) Show that if A and B are similar, then $\det(A) = \det(B)$.

5. Compactness (15 points)

- (a) Let $A \subset \mathbb{R}^n$. Give the “open cover” definition saying what it means for A to be *compact*.
- (b) Show that if $A \subset \mathbb{R}^n$ is compact and $B \subset A$ is closed, then B is compact.
- (c) State the Heine-Borel Theorem.

6. Boundary of a set (10 points)

- (a) Let $A \subset \mathbb{R}^n$. Define ∂A , the *boundary* of A .
- (b) Show that if A is open, then $\partial A \cap A = \emptyset$.

7. Inner Products (10 points)

- (a) Let V be a vector space. Give the general definition for an *inner product* on V .
- (b) Show that $\langle A, B \rangle = \text{tr}(A^t B)$ defines an inner product on $M_n(F)$, where $\text{tr}(C)$ is the *trace* of the matrix C .

8. Dual Spaces (0 points)

*This is just something to think about in case you have already finished the rest of the exam. It is **not** extra credit because I will not be grading it, no matter what you write!*

Let V be a finite-dimensional vector space over F with $\dim(V) = n$. We define

$$V^* = \{f : V \longrightarrow F \mid f \text{ is linear}\},$$

to be the collection of all linear forms on V , known as the **dual** of V . It is easy to verify that V^* is also a vector space over F with addition and scalar multiplication defined in the usual manner: that is, if $f, g \in V^*$ and $\mathbf{v} \in V$ and $a, b \in F$, then $(af + bg)(\mathbf{v}) = a \cdot f(\mathbf{v}) + b \cdot g(\mathbf{v})$.

- (a) Find the dimension of V^* over F , and prove your result.
- (b) Show that $(V^*)^* \cong V$ in a *natural* way.