Last Name: ______________________________

First Name: ______________________________

MATH 23a, FALL 2003
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Final Exam
January 24, 2004

Directions: You have three hours for this exam, though I have designed it to take less than the full amount of time. No calculators, notes, books, etc. are allowed. Please answer on the pages provided–there are blank pages included for scratch work. Note that not all questions (and not all parts) are equally weighted.

Problem	Points	Score
1	44	
2	16	
3	12	
4	10	
5	20	
6	20	
7∗	0	
Total	122	

(*) Please note that question #7 is for your amusement only in case you finish with time to spare.

1. True or False (44 points, 2 each)

T or **F** There exist non-trivial vector spaces with only finitely many vectors.

T or **F** If $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation and $n > m$, then $Ker(L)$ is non-trivial.

T or **F** If $A : U \longrightarrow V$ and $B : V \longrightarrow W$ are both injective linear maps, then $B \circ A : U \longrightarrow W$ is also injective.

T or **F** If U and V are subspaces of a vector space, then $(U + V)/V \cong U/(U \cap V)$.

T or **F** Every skew-symmetric multilinear form $f : V^n \longrightarrow F$ is alternating.

T or **F** If $dim(V) = m$ and $f : V^n \longrightarrow F$ is an alternating form with $n > m$, then $f = 0$.

T or **F** If $A \in O_n(\mathbb{R})$, then $det(A) = \pm 1$.

T or **F** Every Cauchy sequence of integers converges to an integer.

T or **F** Every bounded set of integers has a least element.

T or **F** Every bounded set of rational numbers has a least element.

T or **F** Every bounded set of real numbers has a least element.

T or **F** If $A, B \in M_n(\mathbb{R})$, then $det(AB) = det(BA)$.

T or **F** If $A, B \in M_n(\mathbb{R})$ are similar, then $det(A) = det(B)$.

T or **F** If $A, B \in M_n(\mathbb{R})$ are similar, then $\mathrm{Spec}(A) = \mathrm{Spec}(B)$.

T or **F** If $\sigma = (123)(456) \in S_7$, then $\mathrm{sgn}(\sigma) = +1$.

T or **F** If $\sigma = (123)(456) \in S_7$, then σ is bijective as a function from the set $X = \{1, 2, 3, 4, 5, 6, 7\}$ to itself.

T or **F** If $\sigma = (123)(456) \in S_7$, then $\sigma = \sigma^{-1}$.

T or **F** Given a linear transformation $A : V \to V$, any set of non-zero eigenvectors for A with distinct eigenvalues is linearly independent.

T or **F** Given an inner-product space V, any set of mutually orthogonal non-zero vectors is linearly independent.

T or **F** If V is a subspace of an inner product space U, then $V \cap V^{\perp} = \{\mathbf{0}\}$.

T or **F** If V is an inner product space and $\mathbf{u}, \mathbf{v} \in V$, then $||\mathbf{u} + \mathbf{v}|| = ||\mathbf{u}|| + ||\mathbf{v}||$.

T or **F** If V is an inner product space and $\mathbf{u}, \mathbf{v} \in V$ satisfy $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

2. Completeness (16 points, 4/4/8)

(a) State the Completeness Axiom for the Real Numbers

(b) Give an example of a Cauchy sequence of rational numbers that does not converge to a rational number.

(c) Let $[a_i, b_i] \subset \mathbb{R}$ be a closed interval for each $i \in \mathbb{N}$. Such a collection of closed intervals is said to be *nested* if

$$[a_{i+1}, b_{i+1}] \subset [a_i, b_i]$$

for each $i \in \mathbb{N}$. Given a collection of nested closed intervals as above, let

$$I = \bigcap_{i=1}^{\infty} [a_i, b_i]$$

be the intersection of these intervals.

Show that $I \neq \emptyset$.

3. Cross Products (12 points, 4/4/4/0)

For $\mathbf{u} = (a, b, c), \mathbf{v} = (x, y, z) \in \mathbb{R}^3$, we define the *cross product* of these vectors to be

$$\mathbf{u} \times \mathbf{v} = (bz - cy, cx - az, ay - bx).$$

(a) Show that the cross product is a bilinear map from $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$.

(b) Show that the cross product is skew-symmetric.

(c) Show that if $\mathbb{R}^3$ is given the usual inner product (the dot product), then $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$.

(d) (Bonus Fact, 0 points)
It is true (but you don't have to prove) that

$$||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| \cdot ||\mathbf{v}|| \cdot \sin\theta,$$

where θ is the angle between the two vectors (in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$).

4. A Symmetric Matrix (10 points, 5 each)

Let $A = \begin{bmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{bmatrix} \in M_3(\mathbb{R})$.

(a) For what (real) values of x is A invertible? Explain.

(b) For what (real) values of x is A orthogonal? Explain.

5. A Jordan Block (20 points, 10/5/5)

We have seen in class that some, but not all, matrices are diagonalizable. For a matrix that is not, one of the next best results along these lines would be to be able to put such a matrix in "Jordan Canonical Form." In this problem, we tackle one of the building blocks for this result.

Let $V = \mathbb{R}^2$, and let $A : V \to V$. Suppose that the characteristic polynomial of A is $p_A(\lambda) = (\alpha - \lambda)^2$.

(a) Show that exactly one of the following possibilities must hold:

- A is diagonalizable. (What is the diagonalized form of A?)
- There is a basis for V with respect to which

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}.$$

In the case when $\alpha = 1$, the linear transformation A is known as a *shear*. (Of course, the characteristic polynomial implies that 1 is the only eigenvalue for a shear.)

For the following, let B be a shear:

(b) Use part (a) and a change of basis to show that $(B - I)^2 = 0$. *(Note that B does not already have the form in part (a).)*

(c) Use part (b) to show that $B\mathbf{x} - \mathbf{x} \in V_1$ for every $\mathbf{x} \in \mathbb{R}^2$, where V_1 is the eigenspace corresponding to the eigenvalue 1.

6. Nilpotent Matrices (20 points, 5 each)

Let $V = \mathbb{R}^n$, and let $A : V \to V$. We say that A is *nilpotent* if there is some $m \in \mathbb{N}$ such that $A^m = 0$, and we say that m is the *degree* of nilpotency if $A^m = 0$ but $A^{(m-1)} \neq 0$.

(a) Let $A : V \to V$ be nilpotent of degree m. Let $\mathbf{v} \in V$ be such that $A^{(m-1)}\mathbf{v} \neq \mathbf{0}$. Show that $\{\mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \ldots, A^{(m-1)}\mathbf{v}\}$ is a linearly independent set in V. Conclude that $m \leq n$.

(b) Exhibit examples of nilpotent matrices of degrees 1,2, and 3.

(c) Let $A : V \to V$ be nilpotent. Find all the eigenvalues of A.

(d) Show that if A is nilpotent, then $I + A$ is invertible.
(If A is nilpotent, then $I + A$ is called *unipotent*.)

7. Un digestif: Invertible Matrices over Finite Fields (0 points)

This is just something to think about in case you have already finished the rest of the exam. It is **not** *extra credit because I will not be grading it, no matter what you write!*

Recall that if $M_n(F)$ is the collection of $n \times n$ matrices with entries from the field F, then we define:

$$GL_n(F) = \{A \in M_n(F) \mid \det(F) \neq 0\}.$$

If $F = \mathbb{Z}/p\mathbb{Z}$, where p is prime, then what is $|GL_n(F)|$, for $n = 1$, 2, and 3? (Recall that $|X|$ is the cardinality of the set X.)

(Hint: Consider the column vectors of an invertible matrix.)