

MATH 23a, FALL 2004  
THEORETICAL LINEAR ALGEBRA  
AND MULTIVARIABLE CALCULUS  
Final Exam Preview Problems

**Directions:** What follow are five questions for you to work on in preparation for our in-class final examination, which will take place on Wednesday, January 19, from 2:15–5:15 P.M. These are longer questions that would be difficult in the environment of an exam with strict time constraints, but if you have seen them before, they should be quite reasonable. You should expect that at least one and possibly two of these questions will appear on the in-class final. The questions may not be taken verbatim from this set, they will be very close, and I reserve the right to include parts from any of these questions on the final, in addition to the one/two that will appear in their entirety.

You are encouraged to work on these problems with each other, with the course assistants, or with me, and you may consult any references you like. Keep in mind, however, that you will need to be able to reproduce whatever ideas you gather from other sources, so the premium is on understanding the solutions.

## 1. Orthogonal Matrices.

Let  $V = \mathbb{R}^n$  be Euclidean space, where the usual inner product may be expressed as  $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^t \mathbf{w}$ , where  $\mathbf{v}$  and  $\mathbf{w}$  are thought of as  $n \times 1$  matrices, in coordinates with respect to the standard basis. In other words, if  $\mathbf{v} = (v_1, \dots, v_n)$  and  $\mathbf{w} = (w_1, \dots, w_n)$ , then

$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + \dots + v_n w_n.$$

Recall that we define the collection of invertible linear transformations, called the *general linear group*, to be:

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A \text{ is invertible}\}.$$

Now, we further define a subgroup of the general linear group, called the *orthogonal group*, to be:

$$O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid A^t A = I\}.$$

- (a) Show that  $A \in O_n(\mathbb{R})$  if and only if  $A$  is inner-product preserving. (See HW #8.4.)
- (b) Show that the columns of any  $A \in O_n(\mathbb{R})$  form an orthonormal basis for  $V$  by showing that:
  - i. The columns of any  $A \in O_n(\mathbb{R})$  are vectors of norm 1.
  - ii. The columns of any  $A \in O_n(\mathbb{R})$  are mutually orthogonal.
- (c) If  $A \in O_n(\mathbb{R})$ , find all possible values of  $\det(A)$ .
- (d) For each possible answer in part (c), find the general form of a matrix in  $O_2(\mathbb{R})$  with that determinant.
- (e) For each possible answer in part (d), find all possible eigenvalues for those matrices in  $O_2(\mathbb{R})$ .

## 2. Cramer's Rule

Consider the vector space  $V = F^n$  and the invertible linear transformation  $A : V \rightarrow V$ . If  $\mathbf{b} \in V$  is some fixed vector, the equation  $A\mathbf{x} = \mathbf{b}$  has a unique solution  $\mathbf{x}$ , given as follows:

$$\text{If } A = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix}, \text{ let } \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

$$\text{where } x_i = (\det A)^{-1} \cdot \det \begin{bmatrix} | & & | & | & | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_{i-1} & \mathbf{b} & \mathbf{v}_{i+1} & \cdots & \mathbf{v}_n \\ | & & | & | & | & & | \end{bmatrix}.$$

Prove and apply this result in the following steps:

(Hint: You might do part (e) first to get a feel for this problem.)

- Write  $\mathbf{b}$  as a linear combination of the columns of  $A$ . (Why can this be done?)
- If  $D : V^n \rightarrow F$  is the non-zero alternating form used to define the determinant, evaluate the expression  $D(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{b}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n)$ , in terms of your linear combination from part (a).
- Show that the vector  $\mathbf{x}$  as defined above satisfies the equation  $A\mathbf{x} = \mathbf{b}$ .
- Show that this  $\mathbf{x}$  is the *unique* solution to the equation  $A\mathbf{x} = \mathbf{b}$ .
- Use Cramer's Rule to solve the system of equations when  $F = \mathbb{R}$ :

$$\begin{array}{rclcl} x & + & 2y & + & 3z & = & 1 \\ & & & + & 4z & = & 0 \\ x & & & - & 6z & = & -1 \end{array}$$

### 3. A Jordan Block

We have seen in class that some, but not all, matrices are diagonalizable. For a matrix that is not, one of the next best results along these lines would be to be able to put such a matrix in “Jordan Canonical Form.” In this problem, we tackle one of the building blocks for this result.

Let  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , and suppose that the characteristic polynomial of  $A$  is  $p_A(\lambda) = (\alpha - \lambda)^2$ . (See HW #6.10.)

(a) Show that exactly one of the following two possibilities must hold:

- $A$  is diagonalizable. (What is the diagonalized form of  $A$ ?)
- There is a basis for  $\mathbb{R}^2$  with respect to which the matrix for  $A$  is  $\begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$ .

In the case when  $\alpha = 1$ , the linear transformation  $A$  is known as a *shear*. (Of course, the characteristic polynomial implies that 1 is the only eigenvalue for a shear.)

For the following, let  $B$  be a shear:

- (b) Use the result from part (a) and a change of basis to show that  $(B - I)^2 = 0$ , as linear transformations.  
(Note that  $B$  is a linear transformation and not a matrix, and hence does not already have the form in part (a).)
- (c) Use part (b) to show that  $B\mathbf{x} - \mathbf{x} \in V_1$  for every  $\mathbf{x} \in \mathbb{R}^2$ , where  $V_1$  is the eigenspace corresponding to the eigenvalue 1.

#### 4. Matrices and Roots of Polynomials

Let  $P$  be a polynomial with real coefficients, and let  $A \in M_n(\mathbb{R})$ .

(a) Show that  $\text{Spec}(P(A)) = P(\text{Spec}(A))$ . *Hints:*

*i. Recall that if  $B : V \rightarrow V$  is a linear map, then  $\text{Spec}(B)$  is the set of eigenvalues of  $B$ .*

*ii. If  $P$  is the polynomial  $P(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$ , then  $P(A) = a_k A^k + a_{k-1} A^{k-1} + \cdots + a_1 A + a_0 I$ .*

*iii. If  $S \subset \mathbb{R}$  is any set, then we use the short-hand  $P(S) = \{P(s) \mid s \in S\}$ .*

(b) Using the above property, show that if  $A$  “satisfies” the polynomial  $P$  in the sense that  $P(A) = 0$ , then every eigenvalue of  $A$  is a root of  $P$ . Give an example to illustrate the fact that not all roots of  $P$  must be eigenvalues for  $A$ .

(c) Given  $A \in M_n(\mathbb{R})$  such that  $A^2 = 3A - 2I$ , prove that there exists a natural number  $p \in \{0, 1, \dots, n\}$  such that  $\det A = 2^p$ .

## 5. “Space-Time”

Let  $V = \mathbb{R}^3$ , and consider the bilinear map  $f : V \times V \rightarrow \mathbb{R}$  given by

$$f(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^t B \mathbf{v}, \quad \text{where } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

This form  $f$  defines a new geometry on  $V$ , which has orthogonality defined by  $\langle \mathbf{u}, \mathbf{v} \rangle = 0$  and length defined by  $\|\mathbf{u}\| = \sqrt{|\langle \mathbf{u}, \mathbf{u} \rangle|}$ .

This geometry differs from Euclidean geometry because we find that there are points/vectors whose behaviors vary according to  $f$ :

- If  $\langle \mathbf{u}, \mathbf{u} \rangle > 0$ , we say that  $\mathbf{u}$  is space-like.
- If  $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ , we say that  $\mathbf{u}$  is light-like.
- If  $\langle \mathbf{u}, \mathbf{u} \rangle < 0$ , we say that  $\mathbf{u}$  is time-like.

Because we still have length and orthogonality, we may still speak of an “orthonormal basis” and the “orthogonal complement” of a subspace  $W \subset V$ , but now defined in terms of  $f$ .

- Show that  $f$  is not an inner product.
- Show that  $f$  is not an alternating form.
- Show that any orthonormal basis for  $V$  must consist of two space-like vectors and one time-like vector.
- Let  $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ , and let  $W = \text{span}\{\mathbf{w}\}$ . Find a basis for  $W^\perp$  and show that  $W \cap W^\perp \neq \{\mathbf{0}\}$ .