MATH 23a, FALL 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment #3
Due: October 22, 2004

Reminder: The in-class midterm will take place on Friday, October 29.
(Starred problems make excellent practice exercises!)

1. Read Sections 3.3–3.5 and 5.1–5.2 of Schneider and Barker, and read Sections 1.4–1.5 of Edwards.

2. (A) Let $V$ be a vector space, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ be a collection of linearly independent vectors. Show that no collection of $n-1$ vectors spans $V$.

3. (*) Let $F = \mathbb{Z}/7\mathbb{Z}$. Let $\mathbf{u} = (1, 0, 6)$ $\mathbf{v} = (1, 2, 1)$, and $\mathbf{w} = (2, 1, 3)$ be three vectors in $F^3$, that is, the set of ordered triples with coordinates in $F$. Find coefficients $a, b, c \in F$ to express the vector $\mathbf{x} = (1, 2, 3)$ as a linear combination $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.

4. (A) Let $S$ be some set with finite cardinality $n$. (For simplicity, you may assume that $S = \{1, 2, \ldots, n\}$.) Find a basis for the vector space of functions $V = \{f : S \to \mathbb{R}\}$, where addition and scalar multiplication are defined as for functions of one real variable.

5. (B) Let $V = \{(a_0, a_1, a_2, \ldots) \mid a_i \in \mathbb{R}\}$ be the vector space of all infinite sequences of real numbers. Let $W$ be the subspace of $V$ consisting of all *arithmetic* sequences. Find a basis for $W$, and determine the dimension of $W$. (*A sequence is arithmetic is there is some constant $c$ such that $a_{n+1} = a_n + c$ for all $n \geq 0$.*)

6. (C) (former problem # 2.6, parts f and g)

   Let $p$ be a fixed prime number, and let $V = (\mathbb{Z}/p\mathbb{Z})^n$ with $n \geq 2$.

   (a) How many distinct one-dimensional subspaces does $V$ have?

   (b) How many distinct two-dimensional subspaces does $V$ have?

7. (*) Show that if $W$ is a subspace of $V$ and $dim(V) < \infty$,

   then $dim(W) \leq dim(V)$. (Part of this problem is showing that $W$ *has* a basis. Do this constructively by choosing vectors successively.)

8. (B) Let $U$ and $W$ be subspaces of a vector space $V$. We define two new subspaces as follows:

$$U + W = \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U, \mathbf{w} \in W\}$$
$$U \cap W = \{\mathbf{v} \in V \mid \mathbf{v} \in U \text{ and } \mathbf{v} \in W\}$$

   (a) (*) Convince yourself that both $U + W$ and $U \cap W$ are, in fact, subspaces of $V$.

   (b) Show that if $dim(V) < \infty$, then

$$dim(U + W) = dim(U) + dim(W) - dim(U \cap W).$$

9. (D) Let $P_n(\mathbb{R}) = \{p(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R}, \forall i\}$ be the vector space of all polynomials of degree less than or equal to $n$. Consider the map $L : P_n(\mathbb{R}) \to \mathbb{R}$ defined by $L(p) = \int_0^1 p(x)\, dx$.

   (a) Show that $L$ is a linear map.

   (b) Determine $Im(L)$, and find a basis.

10. (*) Let $L : V \to W$ be a linear map. Show that $L$ is injective if and only if $Ker(L) = \{\mathbf{0}\}$.

11. (E) In this problem, we consider the *shift* operator. Consider the linear map $S : \mathbb{R}^3 \to \mathbb{R}^3$ which acts as follows:

$$S(x, y, z) = (0, x, y).$$

Find the kernel and image of $S$, and verify that

$$dim(Ker(S)) + dim(Im(S)) = dim(\mathbb{R}^3).$$

12. (E) We generalize the notion of the *shift* operator. Let $V$ be the vector space of all infinite sequences of real numbers as in Problem #2.11 and # 3.5 above, and consider the linear maps $S : V \to V$ and $T : V \to V$, where $S$ and $T$ act as follows:

$$S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)$$

$$T(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots)$$

   (a) Find the kernel and image of $S$. How does the result about the dimensions of kernels and images apply?

   (b) Show that $T \circ S = I$ but that $S \circ T \neq I$, where $I : V \longrightarrow V$ is the identity map.

   (c) Which of $S$ and $T$ is onto? Which is one-to-one? Which is invertible? Explain.

13. (D) Consider the linear differential operator $D : C^\infty \longrightarrow C^\infty$ given by $D(f) = f' + af$, where $a$ is some fixed real number. (Recall that $C^\infty$ is the vector space of all functions which are infinitely-differentiable.)

    (a) Find $Ker(D)$.

    (b) Show that $D$ is surjective.
    (Hints: 1. Let $g \in C^\infty$, and show that $f' + af = g$ has a solution. Multiply both sides by the *integrating factor* $e^{ax}$, and integrate both sides, using the product rule on the left-hand side! 2. Cite an appropriate theorem from Calculus to justify the existence of an anti-derivative.)

14. (*) Let $A : U \rightarrow V$ and $B : V \rightarrow W$ be linear maps between finite-dimensional vector spaces.

    (a) Show that if $A$ and $B$ are both surjective, then so is $B \circ A$.

    (b) Show that if $B$ is one-to-one, then $Ker(B \circ A) = Ker(A)$.

    (c) Assuming $A$ and $B$ are surjective, show that $dim(Ker(B \circ A)) = dim(Ker(A)) + dim(Ker(B))$.

15. (*) Find the inverse of the linear map $L : (\mathbb{Z}/7\mathbb{Z})^3 \longrightarrow (\mathbb{Z}/7\mathbb{Z})^3$ given by $L(x, y, z) = (x + y + z, 2x + 3y + 4z, 3x + 4y + 6z)$.

16. (*) Let $F$ be any field. Show that the linear operator $L : F^2 \longrightarrow F^2$ given by $L(x, y) = (ax+by, cx+dy)$ is invertible if and only if $ad-bc \neq 0$.

17. (*) Show that if $A : U \longrightarrow V$ and $B : V \longrightarrow W$ are both invertible linear maps, then $(BA)^{-1} = A^{-1}B^{-1}$.