MATH 23a, FALL 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment #5
Due: November 12, 2004

1. Read Chapter 1 (especially Sections 1.1–1.4) and Sections 3.5, 5.2, and 5.3 from Schneider and Barker, and Section 1.4 of Edwards for a short summary.

2. (A) Suppose λ is an eigenvalue for the linear transformation $A : V \longrightarrow V$.

   (a) Show that λ^2 is an eigenvalue for A^2.

   (b) If A is invertible, show that λ^{-1} is an eigenvalue for A^{-1}. (What happens if $\lambda = 0$?)

3. (*) Find examples of *invertible* linear transformations $A : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ such that:

   (a) A has no eigenvalues.

   (b) A has only one eigenvalue λ, but $dim(V_\lambda) < 4$.

   (c) $\mathbf{e}_1 = (1, 0, 0, 0)$ and $\mathbf{v} = (1, 1, 1, 1)$ are both eigenvectors but have distinct eigenvalues.

4. (B) Let $A : V \to V$ be linear, and let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be non-zero eigenvectors for A with *distinct* eigenvalues $\lambda_1, \ldots, \lambda_m$, respectively. Show that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly independent.

5. (C) Two linear transformations $A : V \longrightarrow V$ and $B : V \longrightarrow V$ are said to be *similar* if there exists an invertible linear transformation $S : V \longrightarrow V$ such that $A = SBS^{-1}$. Consider the following, where a well-written answer may suffice for both parts:

   (a) Show that if A and B are similar, then $Spec(A) = Spec(B)$. (Recall that $Spec(A)$ is defined to be the set of eigenvalues for A. In this problem, to show that $Spec(A) = Spec(B)$, we show that if λ is an eigenvalue for A, then λ is an eigenvalue for B, and vice versa.)

   (b) Suppose A and B are similar and λ is an eigenvalue for both. Find the precise relationship between the eigenspace for λ with respect to A and the eigenspace for λ with respect to B.

6. (D) Let P_3 be the vector space of polynomials of degree less than or equal to 3, with real coefficients.

   Let $\mathfrak{B}_1 = \{1, x, x^2, x^3\}$, $\mathfrak{B}_2 = \{1, 1+x, 1+x^2, 1+x^3\}$, and $\mathfrak{B}_3 = \{1+x, 1-x, x^2-x^3, x^2+x^3\}$ be bases for P_3. Let $D : P_3 \longrightarrow P_3$ be the usual differentiation operator, and let $I : P_3 \longrightarrow P_3$ be the identity.

   (a) For each of the three bases, write down the matrix for D with respect to that basis (with the basis in question considered as the basis for both the domain and range).

   (b) Write down the matrix for I where the domain has basis $\mathfrak{B}_1$ and the range has basis $\mathfrak{B}_2$.

7. (E) Let $V = \mathbb{R}^2$ be two-dimensional Euclidean space, with its usual x- and y- coordinate axes. Consider the linear transformation $L_\alpha : V \longrightarrow V$ that performs a reflection about the line $y = \alpha x$.

   (a) Write the matrix for L_α with respect to the basis $\mathfrak{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$. (Hint: Use elementary geometry to compute $L_\alpha(\mathbf{e}_1)$ and $L_\alpha(\mathbf{e}_2)$.)

   (b) Calculate the matrix for $L_\beta \circ L_\alpha$ (with respect to $\mathfrak{B}$) in two ways: by multiplying the matrices for L_β and L_α, and by determining the matrix for the resulting composed linear transformation directly.

   (c) Show that the composed linear transformation $L_\beta \circ L_\alpha$ is a rotation. By what angle are vectors in $\mathbb{R}^2$ rotated under this transformation?

8. (*) Read pp. 30–31 from Schneider and Barker for the definition of the *transpose* of a matrix. Check that their definition is equivalent to the following:

   Definition. If $A = [a_{ij}]$ is an $n \times m$ matrix then we define its **transpose** $A^t = [a_{ji}]$ to be the $m \times n$ matrix whose rows are the columns of A. That is,

$$\text{if } A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \quad \text{then } A^t = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1m} & \cdots & a_{mn} \end{bmatrix}.$$

   Note that this definition applies equally well to vectors, which may be considered as $n \times 1$ matrices. Finally, read the three most important theorems concerning transposed matrices, and convince yourself of their validity.

   Theorem: If A and B are $n \times m$, then $(A+B)^t = A^t + B^t$.

   Theorem: If A is $n \times m$ and B is $m \times k$, then $(AB)^t = B^t A^t$.