Last Name: ______________________________

First Name: ______________________________

MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm (in-class portion)
October 30, 2002

Directions: You have one hour for this exam. No calculators, notes, books, etc. are allowed. Please answer on the pages provided. Show all work!

Problem	Points	per part	Score
1	23	1 each	
2	24	3 each	
3	15	5/10	
4	20	10/5/5	
5	20	5/5/10	
6	18	6 each	
Total	120	120	

1. **Properties of Number Systems**

   For each entry in the following table, identify whether the number system at the head of the column has that property or not by placing a "Y" for yes and an "N" for no.

Property	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{Z}/6\mathbb{Z}$
Existence of Additive Inverses					
Existence of Multiplicative Inverses					
Ordered					
Well-ordered					
Complete	X				X

2. **True or False**

T or **F** If $L : V \longrightarrow V$ is an injective linear map, then it is surjective.

T or **F** If $L : V \longrightarrow W$ is a bijective linear map, then $V \cong W$.

T or **F** If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is a set of linearly independent vectors, then any $\mathbf{v} \in V$ may be written as a linear combination of these vectors in a unique way.

T or **F** If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is a set of vectors such that none of them is a scalar multiple of any of the others, then the set is linearly independent.

T or **F** If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is a set of vectors such that $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = V$, then $dim(V) \leq n$.

T or **F** If $L : V \longrightarrow V$ is a linear transformation, then $Ker(L)$ is a subspace of V.

T or **F** The set of vectors $\{(0, 1, 2), (1, 1, 1), (0, 0, 0)\}$ is linearly independent in $\mathbb{R}^3$.

T or **F** If $V = \text{span}\{(0, 1, 2, 3, 4), (1, 1, 1, 1, 1)\}$, then $dim(V) = 5$.

3. Let $\{a_n\}$ and $\{b_n\}$ be sequences of rational numbers.

   (a) Define what it means for $\{a_n\}$ to be a **Cauchy sequence**.

   (b) If $\{a_n\}$ and $\{b_n\}$ are both Cauchy sequences, show that $\{a_n \cdot b_n\}$ (the term-by-term product of the original two sequences) is also a Cauchy sequence.

4. Let $T : V \to V$ be a non-zero linear map. We consider the situation when $T^2 = 0$.

   (Note that $T^2 = T \circ T$, in other words, T composed with itself, and 0 is the zero linear map that takes every vector in V to $\mathbf{0}$.)

   (a) Show that $T^2 = 0$ if and only if $Im(T) \subset Ker(T)$.

   (b) Show that if $T^2 = 0$ and $dim(V) = 3$, then $dim(Ker(T)) = 2$.

   (c) Give an example of a non-trivial vector space V and a non-trivial linear map $T : V \to V$ satisfying $T^2 = 0$.

5. Let $P_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 | a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ be the vector space of polynomials of degree three or less with real coefficients. Consider the linear map $I : P_3 \longrightarrow \mathbb{R}$ defined by $I(p(x)) = \int_0^1 p(x)\, dx$, for $p(x) \in P_3$.

   (a) Find $dim(Ker(I))$.

   (b) Exhibit a basis for $Ker(I)$.

   (c) Prove that your answer to part (b) is in fact a basis.

6. Let V be a vector space over the field F, and let W be a subspace of V. We define the **quotient space** V/W as follows:

$$V/W = \{\mathbf{v} + W | \mathbf{v} \in V\}/\sim,$$

where $\mathbf{v}_1 + W \sim \mathbf{v}_2 + W$ if and only if $\mathbf{v}_1 - \mathbf{v}_2 \in W$.

The elements of the quotient space are called **cosets**, and they have the form

$$\mathbf{v} + W = \{\mathbf{v} + \mathbf{w} | \mathbf{w} \in W\}.$$

If we define addition and scalar muliplication as follows:

$$(\mathbf{v}_1 + W) + (\mathbf{v}_2 + W) = (\mathbf{v}_1 + \mathbf{v}_2) + W$$

$$c \cdot (\mathbf{v} + W) = (c \cdot \mathbf{v}) + W$$

for any $c \in F$ and any $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in V$, then in fact, the quotient space V/W is a vector space over F.

(a) It is a fact that $\sim$ is an equivalence relation. Show that $\sim$ is *transitive.*

(b) It is a fact that addition and scalar multiplication are well-defined. Show that *addition* is well-defined.

(c) It is a fact that V/W is a vector space over F. Show that V/W satisfies axiom V3 concerning *additive identities.*