

MATH 23a, FALL 2004
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Midterm Solutions (in-class portion)
October 29, 2004

1. True or False

The successor of any natural number is another natural number.

True. This is Peano's First Postulate.

The integers form an ordered field.

False. They are not a field, because all but 1 and -1 lack multiplicative inverses.

The rational numbers are well-ordered.

False. They are not well-ordered because there are non-empty sets, such as $\{x \in \mathbb{Q} \mid 0 < x < 1\}$, that lack smallest elements.

$\mathbb{Z}/n\mathbb{Z}$ is a commutative ring for any integer $n \geq 2$.

True.

In any ordered ring with identity, $0 < 1$.

True.

According to the equivalence relation that we used in class to define the integers, the two pairs of natural numbers, $(1, 3)$ and $(2, 6)$, are equivalent.

False. Two pairs (a, b) and (c, d) are equivalent if $a + d = b + c$.

A field is a vector space over itself.

True. In fact, a one-dimensional vector space.

The set $\{0\}$ is a vector space over any field.

True.

The vector space \mathbb{R}^2 is a subspace of \mathbb{R}^3 .

False. It is not a subset of \mathbb{R}^3 and hence cannot be a subspace. It is *isomorphic* to a subspace of \mathbb{R}^3 , but this is not the same thing.

If $V = (\mathbb{Z}/p\mathbb{Z})^n$ with p prime and $n \geq 2$, then V has $(p^n - 1)/(p - 1)$ distinct 1-dimensional subspaces.

True. See homework set 3.

If $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\dim(V) = 2$.

False. The two vectors may not be linearly independent.

If $\dim(V) = n$, then any set of $n + 1$ (or more) vectors in V must span V .

False. Again, the vectors may not be linearly independent.

The set of functions $\{1, \cos x, \cos 2x\}$ is linearly independent, considered as a subset of $C[0, 2\pi]$.

True.

If a set of vectors spans V , then that set contains a basis for V .

True. This is a fact from class.

If $L : V \rightarrow W$ is linear and injective, then $\dim(\text{Im}(L)) = \dim(V)$.

True. This follows from the Rank-Nullity Theorem. Since L is injective, $\text{Ker}(L) = \{\mathbf{0}\}$ and hence $\dim(\text{Ker}(L)) = 0$.

If $L : V \rightarrow W$ is linear and surjective, then L is invertible.

False. L may not be injective.

If $L : V \rightarrow W$ is linear and bijective, then $V \cong W$.

True. This is the definition of isomorphic.

If $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a linearly dependent set in the vector space U , and $W = \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$, then $\dim(W) < n$.

True.

$P_3(\mathbb{R}) \cong \mathbb{R}^4$, where $P_3(\mathbb{R})$ is the collection of polynomials with real coefficients and degree less than or equal to 3.

True. Each of these is a 4-dimensional real vector space.

2. Let V be a vector space over the field F , and let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for V . Let $\mathbf{w} \in V$ be a randomly chosen vector. Prove that there exists a unique way to write \mathbf{w} as a linear combination of the vectors in the basis.

Since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V , it spans V . By the definition of span, any vector $\mathbf{w} \in V$ may be written as a linear combination of the basis elements, say as:

$$\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \quad (1)$$

We now show that this expression is unique. Suppose there is another linear combination of the basis elements that gives \mathbf{w} , say:

$$\mathbf{w} = d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n \quad (2)$$

Subtracting equation (2) from equation (1) yields:

$$\mathbf{0} = (c_1 - d_1)\mathbf{v}_1 + \cdots + (c_n - d_n)\mathbf{v}_n \quad (3)$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V , it is a linearly independent set, and hence all of the coefficients in equation (3) must be 0. In other words, $c_1 = d_1$, $c_2 = d_2$, and so on up through $c_n = d_n$, and the two expressions in equations (1) and (2) are identical.

3. Consider the field $F = \mathbb{Z}/7\mathbb{Z}$ and the vector space $V = (\mathbb{Z}/7\mathbb{Z})^2$. Let $L : V \rightarrow V$ be the linear map given by:

$$L(x, y) = (2x + 5y, x + 6y)$$

(For simplicity throughout the problem, you may denote the elements of F by 0, 1, 2, 3, 4, 5, and 6 rather than with the usual equivalence class notation.)

- (a) Write down *all* of the subspaces of V .

0-dimensional: $\{\mathbf{0}\}$
 1-dimensional: $\text{span}\{(0, 1)\}$ $\text{span}\{(1, 0)\}$
 $\text{span}\{(1, 1)\}$ $\text{span}\{(1, 2)\}$
 $\text{span}\{(1, 3)\}$ $\text{span}\{(1, 4)\}$
 $\text{span}\{(1, 5)\}$ $\text{span}\{(1, 6)\}$
 2-dimensional: V

(b) Identify $Ker(L)$ as one of the subspaces of V from part (a).

The vector $(x, y) \in Ker(L)$ if and only if $L(x, y) = (0, 0)$. In other words, we are looking for all vectors (x, y) such that $(2x + 5y, x + 6y) = (0, 0)$. To say that two ordered pairs are equal means that each of the corresponding terms are equal, and so this becomes the pair of equations:

$$2x + 5y = 0 \quad \text{and} \quad x + 6y = 0.$$

Keeping in mind that we are doing arithmetic in the finite field $F = \mathbb{Z}/7\mathbb{Z}$, we re-arrange the second equation to discover that $x = y$ (since $-6 = 1$). Plugging this into the first equation gives $7y = 0$, but this is the tautology $0 = 0$ (since $7 = 0$). In other words, the only condition for (x, y) to be in $Ker(L)$ is that $x = y$. If this vector is of the form (x, x) , it may be re-written as $x(1, 1)$. That is, we have $Ker(L) = span\{(1, 1)\}$.

(c) Identify $Im(L)$ as one of the subspaces of V from part (a).

The image of L is precisely the set of vectors of the form $(2x + 5y, x + 6y)$. Working in the arithmetic of $F = \mathbb{Z}/7\mathbb{Z}$, we re-write this vector as follows:

$$\begin{aligned} (2x + 5y, x + 6y) &= x(2, 1) + y(5, 6) \\ &= x(2, 1) + y(-2, -1) \\ &= x(2, 1) - y(2, 1) \\ &= (x - y)(2, 1) \end{aligned}$$

Since x and y are free to vary, we see that $x - y$ may be any scalar in F , and we get $Im(L) = span\{(2, 1)\}$.

(d) What do the results of parts (b) and/or (c) say about the invertibility of L ?

In part (b), we observed that $Ker(L)$ is non-trivial. A result from the homework says that L is injective if and only if $Ker(L)$ is trivial, and hence L is not injective, one of the conditions for being invertible.

On the other hand, part (c) shows that $dim(Im(L)) = 1$, while $dim(V) = 2$, and hence L is not surjective, the other condition for being invertible.

4. On the homework (problem #3.4), we considered the vector space of real-valued functions whose domain was a finite set S and showed that the dimension of this vector space was equal to the cardinality of S . That is, if $S = \{a_1, a_2, \dots, a_n\}$ and $V = \{f : S \rightarrow \mathbb{R}\}$, then $\dim(V) = n$.

For this problem, consider the special case $S = \{a, b, c, d, e\}$, and once again let $V = \{f : S \rightarrow \mathbb{R}\}$. Now, let

$$W = \{f \in V \mid f(a) + f(b) = 0 \text{ and } f(d) - f(e) = 0\}.$$

- (a) Show that W is a subspace of V .

It suffices to show that W is closed under the two basic vector space operations:

- i. Vector addition:

Let f and g be functions in W . We show that $f + g$ is also in W by checking the two conditions for membership in W . First,

$$(f+g)(a)+(f+g)(b) = [f(a)+g(a)]+[f(b)+g(b)] = [f(a)+f(b)]+[g(a)+g(b)] = 0,$$

where we have used the definition of $f + g$, the associativity of addition in \mathbb{R} , and the fact that $f, g \in W$ in the successive steps. Second,

$$(f+g)(d)-(f+g)(e) = [f(d)+g(d)]-[f(e)+g(e)] = [f(d)-f(e)]+[g(d)-g(e)] = 0,$$

which uses the same justifications as the previous series of equations.

- ii. Scalar multiplication:

Let $f \in W$, and let $c \in F$. We show that $c \cdot f$ is also in W . First,

$$(c \cdot f)(a)+(c \cdot f)(b) = [c \cdot f(a)]+[c \cdot f(b)] = c \cdot [f(a)+f(b)] = 0,$$

and second,

$$(c \cdot f)(d)-(c \cdot f)(e) = [c \cdot f(d)]-[c \cdot f(e)] = c \cdot [f(d)-f(e)] = 0,$$

for reasons similar to those in the vector addition part.

(b) Write down a basis for W .

A basis will consist of functions in W , and any such function will be determined by its values on the domain S . Consider the set $\mathfrak{B} = \{f_1, f_2, f_3\}$, where:

$$\begin{array}{cccccc} f_1(a) = 1 & f_1(b) = -1 & f_1(c) = 0 & f_1(d) = 0 & f_1(e) = 0 \\ f_2(a) = 0 & f_2(b) = 0 & f_2(c) = 1 & f_2(d) = 0 & f_2(e) = 0 \\ f_3(a) = 0 & f_3(b) = 0 & f_3(c) = 0 & f_3(d) = 1 & f_3(e) = 1 \end{array}$$

(c) Show that your answer to part (b) is, in fact, a basis for W .

First, we show that our basis \mathfrak{B} spans W . Let g be any function W , determined by the following values:

$$g(a) = k_1 \quad g(b) = k_2 \quad g(c) = k_3 \quad g(d) = k_4 \quad g(e) = k_5$$

Then, by the conditions for being in W , we know that $g(a) + g(b) = 0$ and that $g(d) - g(e) = 0$. In other words, $k_2 = -k_1$ and $k_5 = k_4$, so that we may re-write the values of g as:

$$g(a) = k_1 \quad g(b) = -k_1 \quad g(c) = k_3 \quad g(d) = k_4 \quad g(e) = k_4$$

This leads us to observe that $g = (k_1 \cdot f_1) + (k_3 \cdot f_2) + (k_4 \cdot f_3)$, and hence these three functions span W .

Next, we show that the functions in \mathfrak{B} are linearly independent. Suppose that $(c_1 \cdot f_1) + (c_2 \cdot f_2) + (c_3 \cdot f_3) = 0$. Then we evaluate these two functions on the domain element a to see that:

$$\begin{aligned} [(c_1 \cdot f_1) + (c_2 \cdot f_2) + (c_3 \cdot f_3)](a) &= 0(a) \\ c_1 \cdot f_1(a) + c_2 \cdot f_2(a) + c_3 \cdot f_3(a) &= 0 \\ c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0 &= 0 \\ c_1 &= 0 \end{aligned}$$

By evaluating at c and d , it is easy to see that $c_2 = 0$ and $c_3 = 0$, respectively, as well, and thus, these functions are linearly independent.

5. In this problem, we consider the First Isomorphism Theorem and its proof.

Theorem. (The First Isomorphism Theorem)

Let $L : V \rightarrow W$ be linear and surjective. Then:

$$V/\text{Ker}(L) \cong W.$$

Of course, we need to explain our notation. Here, $V/\text{Ker}(L)$ is the set of equivalence classes of vectors in V , with the equivalence relation \sim defined as follows:

$$\mathbf{u} \sim \mathbf{v} \text{ provided that } \mathbf{u} - \mathbf{v} \in \text{Ker}(L)$$

In other words, an equivalence class $[\mathbf{u}] \in V/\text{Ker}(L)$ is a set of the form:

$$[\mathbf{u}] = \{\mathbf{v} \in V \mid \mathbf{u} - \mathbf{v} \in \text{Ker}(L)\}$$

We will show in class next week that, in fact, $V/\text{Ker}(L)$ is a vector space, but for this problem, you may assume that it is.

Proof:

Define $\varphi : V/\text{Ker}(L) \rightarrow W$ as follows:

$$\varphi([\mathbf{u}]) = L(\mathbf{u})$$

(a) Show that φ is well-defined.

Suppose $[\mathbf{u}] = [\mathbf{u}']$, that is, that $\mathbf{u} \sim \mathbf{u}'$. By definition of the equivalence $\mathbf{u} - \mathbf{u}' \in \text{Ker}(L)$. Then:

$$\begin{aligned} \varphi([\mathbf{u}']) &= L(\mathbf{u}') && \text{by the definition of } \varphi \\ &= L(\mathbf{u}') + \mathbf{0} && \text{by axiom V3 in vector spaces} \\ &= L(\mathbf{u}') + L(\mathbf{u} - \mathbf{u}') && \text{since } \mathbf{u} - \mathbf{u}' \in \text{Ker}(L) \\ &= L(\mathbf{u}' + (\mathbf{u} - \mathbf{u}')) && \text{since } L \text{ is linear} \\ &= L(\mathbf{u}) && \text{by axioms V1-V4 in vector spaces} \\ &= \varphi([\mathbf{u}]) && \text{by the definition of } \varphi \end{aligned}$$

and hence φ is well-defined.

(b) We showed in class that φ is linear.

(c) Show that φ is surjective.

Let $\mathbf{w} \in W$. Since L is surjective, there exists some $\mathbf{u} \in V$ such that $L(\mathbf{u}) = \mathbf{w}$. Note then that $\varphi([\mathbf{u}]) = L(\mathbf{u}) = \mathbf{w}$, and so $[\mathbf{u}]$ is the element in $V/\text{Ker}(L)$ that is mapped to \mathbf{w} .

(d) Show that φ is injective.

Suppose $\varphi([\mathbf{u}]) = \varphi([\mathbf{v}])$. Then, by definition of φ , we have $L(\mathbf{u}) = L(\mathbf{v})$, or in other words, $L(\mathbf{u}) - L(\mathbf{v}) = \mathbf{0}$. Since L is linear, this becomes $L(\mathbf{u} - \mathbf{v}) = \mathbf{0}$. But this means that $\mathbf{u} - \mathbf{v} \in \text{Ker}(L)$, or in other words, $\mathbf{u} \sim \mathbf{v}$, according to the definition of the equivalence. Lastly, this implies that $[\mathbf{u}] = [\mathbf{v}]$, and so φ is injective.