

5. Short Exact Sequences

Let U , V , and W be vector spaces (not necessarily finite-dimensional) over the same field, and suppose we have the following four linear maps:

$$A : \{0\} \longrightarrow U$$

$$S : U \longrightarrow V$$

$$T : V \longrightarrow W$$

$$B : W \longrightarrow \{0\}$$

with the property that the kernel of any of them is equal to the image of the previous one. In other words, $\text{Ker}(S) = \text{Im}(A)$, $\text{Ker}(T) = \text{Im}(S)$, and $\text{Ker}(B) = \text{Im}(T)$.

Such a collection of vector spaces and linear maps is known as a (*short exact sequence*), which is often denoted by:

$$\{0\} \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow \{0\}$$

- (a) Show that S is injective.
- (b) Show that T is surjective.
- (c1) Show that $U \cong S(U)$.
- (c2) Show that $W \cong V/U$.
(Note that this is an abuse of notation since U is not, strictly speaking, a subspace of V . We are using part (c1) to identify U with its image $S(U)$.)
- (d) Consider the case when $V = C^\infty(\mathbb{R})$, the vector space of all infinitely differentiable real-valued functions, and the map T is differentiation, that is $T(f) = f'$. What choice of a vector space U and a map S make $\{0\} \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow \{0\}$ into an exact sequence? (Bonus: What subspace of $C^\infty(\mathbb{R})$ is W in this case?)