A Note on Proofs

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I originally wrote this note in the fall of 2002 for the benefit of the students taking Math 23a at Harvard University, who were under the excellent instruction of John Boller. Math 23 is a first-year class introducing rigorous calculus and linear algebra, and was designed to be the first experience most of its students had with pure math, and in particular, with proofs. Being a course assistant and grader, I wrote this note from a grader's perspective, immediately after having graded the first homework assignment.[1] It addresses the more important things one must do when writing proofs, and points out the most common mistakes. If you've never written a mathematical proof before (and have some desire to do so), then this note is for you.

A proof, at its base, is an argument. I assume that you know how to make persuasive arguments — everybody has had late-night philosophical debates with roommates. If you are asked to prove, for instance, that there are an infinite number of prime numbers, then your job is to convince the reader of the truth of that statement. Your proof should read like any other written argument: it should have a thesis, it should have a logical progression, and it should be in grammatically-correct English.[2] But there is an essential difference between a mathematical proof and a philosophy paper: a mathematical proof should be so precise that there is (theoretically) no room for error. None. That's the beautiful thing about pure mathematics: it's the only subject in which you can be absolutely sure of the truth of your statements. It's not enough to find overwhelming evidence for a statement in order to prove its truth. You can leave no room to doubt that what you say is completely correct.

The proof then, were you to write it out in full, would be extremely long-winded, because every step must be meticulously exact. This is where mathematical notation comes in — common precise constructions like "for all x in the set A" can be shortened to "$\forall x \in A$." This is the first important point about writing proofs: were you to expand out all of the notation (e.g., replace each "$\exists x$" with "there exists an x"), *you should have a grammatically correct paper.*

As with any argument, a proof is a logical path from a starting point to an ending point. So in order to write one, the first thing to do is start with a *precise*

[1] I then threw that version away and rewrote it the next morning so that I would sound helpful instead of frustrated.

[2] Unless your class is being given in another language, in which case it's bizarre that you're reading this note.

statement of your assumptions and work towards a *precisely stated conclusion*. By precise, I mean something totally unambiguous, with nothing left for interpretation. For example, the prime factorization theorem might be stated "Every $n \in \mathbf{N}$ can be written as a product $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, where $p_1, \ldots, p_m$ are distinct prime numbers and $a_1, \ldots, a_m$ are natural numbers." For more examples, read the statements of the lemmas and theorems in your textbooks.

The only mathematical way to get from the assumptions to the conclusions is by making *precise logical deductions*, that is, statements that use facts that you know or have already shown and imply other facts. Each statement should again be totally unambiguous, and have impeccable support: you must be able to justify each statement with a mathematical reason that cannot be argued, like a theorem out of a book (e.g., if p is prime and n is any natural number then there exists a nonegative integer m such that p^m divides n but p^{m+1} does not, *because of the existence of prime factorizations*). There cannot be any room for the reader to say "but what if" or "now why is that" — every statement must be infallible. Again, this is the beauty of mathematics — whereas one rarely wins a philosophical debate with one's roomates (especially at three in the morning), the correctness of a mathematical argument cannot be debated. It's just true.

This does not mean that you have to write down *every* step; this would make your proofs impossibly long, even with generous use of math notation. For instance, you do not have to cite the distributive and associative properties of the real numbers to say that $(a+b)(c+d) = ac + ad + bc + bd$ (but you should know how to prove it, in case someone doesn't believe that step). In each case, use your judgment about whether the logical leap you are about to make is small enough that it does not require further justification. If done well, this will make your proofs much less tedious, while still not leaving any of your statements up for debate; you simply leave out few enough logical steps that the reader can easily fill them in. (At this point, however, what you should consider is whether the grader will believe that *you* know how to fill them in.)

There are also several things to avoid when writing a proof. One of the most common mistakes is to write a proof by example. A "proof" by example is *not* a proof. Examples are never necessary in a proof, and are only relevent if there are only a finite number of cases and you prove them all. For instance, if you prove that 10 has a prime factorization $2 \cdot 5$, then that's great, but you haven't proved that 12 has a prime factorization too. If your proof for 10 generalizes to any natural number, then write the general proof; the specific case will follow. It is often helpful to work out an example if you don't know how to prove something in general, but you still need to do the general proof afterwards. So you can usually omit any examples — if you find yourself needing to insert them for clarity, then you should usually try to make your general proof more clear. Adding examples when it's not necessary can serve to clutter your proofs and confuse your reader, at least until you're more fluent at proof-writing. It is likely, as you take more advanced classes and have to write harder proofs, that you will run into situations where an abstract definition or a statement would become much more concrete if an example is added; in that case, an example may be justified, to the extent that it helps your reader understand the *general* definition or proof, which is the importart part. For the moment, it's probably best to leave out examples until you're confident in your proof-writing abilities.

The same rules apply to writing down your intuition for a proof — it's never necessary for the proof, and it is often best to omit it. I don't mean to say that intuition is unimportant; on the contrary, mathematical intuition is the absolute most valuable thing a mathematician can have; it's what makes the scrawls on the blackboard concrete to you. However, it is a means to an end, and the end is a proof of something. It may be helpful for the reader if you wrote something like, "intuitively, one would think that $x_n \to 0$, so we will work towards proving that," and that's probably the extent of what you should include in your proofs, at least at a beginning level. Again, it is perhaps best to leave out heuristics altogether at this point, just so there is no chance of you or your reader confusing them with logical deductions.

Another common mistake to avoid is circular reasoning. Be very careful that you don't at some point during a proof implicitly assume a result that follows from what you are trying to prove. For instance, if you are going to prove that any integer n is divisible by some prime p, you cannot assume the existence of prime factorizations, since the former is used to prove the latter.

In math, the easiest way to confuse a reader is to use a variable that you didn't define. *Always* define your variables *before* you first use them, or at the very latest, later in the same sentence.

Lastly, always remember that the reason you are writing a proof is so that someone else can read it. When you write a proof, read over it through the eyes of someone who's not a mathematical genius, and who doesn't have any idea what you're talking about before your proof begins. If that person gets confused, you need be more clear.

I hope you find this helpful.