Solution Set 1B

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Math 23a
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(B) Verify that multiplication is well-defined for integers, as defined by equivalence classes of ordered pairs of natural numbers.

Solution: Given two equivalence classes of ordered pairs of natural numbers, $\{(a, b)\}$ and $\{(c, d)\}$ we define multiplication as:

$$\{(a, b)\} * \{(c, d)\} = \{(ad + bc, ac + bd)\}.$$

In order to prove that this is a well-defined operation, assume that (a, b) is equivalent to (a', b'); in other words, $\{(a, b)\} = \{(a', b')\}$ for $a, b, a', b' \in \mathbb{N}$.

We must show that if we take some generic integer $\{(c, d)\}$, with $c, d \in \mathbb{N}$,
$\{(a, b)\} * \{(c, d)\} = \{(a', b')\} * \{(c, d)\}$.

Hence, we need to show that $\{(ad + bc, ac + bd)\} = \{(a'd + b'c, a'c + b'd)\}$.

Proof: By our equivalence relation, we have that $a + b' = b + a'$.
Noting that multiplication is well-defined for natural numbers,
we can conclude that $c(a + b') = c(b + a')$ and that $d(b + a') = d(a + b')$.

Hence: $ac + b'c = bc + a'c$ and $bd + a'd = ad + b'd$.

Since addition is also well-defined for natural numbers, we can add these two equations, leading us to conclude that $a'd + b'c + ac + bd = ad + bc + a'c + b'd$.
Grouping suggestively using associativity, we thus have that:

$$(a'd + b'c) + (ac + bd) = (ad + bc) + (a'c + b'd).$$

Hence, $(ad + bc, ac + bd)$ is equivalent to $(a'd + b'c, a'c + b'd)$, and so,

$$\{(ad + bc, ac + bd)\} = \{(a'd + b'c, a'c + b'd)\}.$$

Thus, $\{(a,b)\} * \{(c,d)\} = \{(a'b')\} * \{(c,d)\}$ as desired.

Notes: A few comments:
1: Many people tried to subtract two natural numbers in their proofs. Remember that the natural numbers do not have additive inverses, so this operation is disallowed.

2: Additionally, asserting the existence of negative natural numbers by invoking injectivity of $+$ is not appropriate. While it is true that, as shown in class, injectivity can sometimes be used to justify a cancellation step, this is significantly different than invoking it to establish the existence of elements that do not necessarily exist.

3: Finally, a few people felt it necessary to prove the case where (a,b) is equivalent to (a',b') **and** (c,d) is equivalent to (c',d'). While thoroughness is certainly encouraged, here it is unnecessarily messy, as we can just extend our logic for the above proof to justify multiplication in this more complicated case.