

## SOLUTION SET 3B

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5. Let  $V = \{(a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{R}\}$  be the vector space of all infinite sequences of real numbers. Let  $W$  be the subspace of  $V$  consisting of all *arithmetic* sequences. Find a basis for  $W$ , and determine the dimension of  $W$ . (A sequence is arithmetic if there is some constant  $c$  such that  $a_{n+1} = a_n + c$  for all  $n \geq 0$ .)

Arithmetic sequences have the form  $(a_0, a_0 + c, a_0 + 2c, \dots)$  where  $a_0 \in \mathbb{R}$  and  $c \in \mathbb{R}$ . There are of course infinitely many ways of choosing a basis, but perhaps the most straightforward one is  $(1, 1, 1, 1, \dots), (0, 1, 2, 3, \dots)$ .

Both  $(1, 1, 1, 1, \dots)$  and  $(0, 1, 2, 3, \dots)$  are arithmetic sequences and so are in  $W$ . Any sequence  $(a_0, a_0 + c, a_0 + 2c, \dots)$  in  $W$  can be expressed as a linear combination of these two vectors:

$$(a_0, a_0 + c, a_0 + 2c, \dots) = a_0(1, 1, 1, \dots) + c(0, 1, 2, \dots)$$

. As linear combinations of these two vectors form all arithmetic sequences in  $V$  and nothing but arithmetic sequences in  $V$ , they span  $W$ . Furthermore, since they are not scalar multiples of each other,  $(1, 1, 1, 1, \dots), (0, 1, 2, 3, \dots)$  is linearly independent and therefore forms a basis for  $W$ .

Since we have a basis for  $W$  consisting of two elements,  $\dim(W) = 2$ . This is consistent with the fact that we need only two pieces of information to specify any arithmetic sequence in  $V$ , i.e. the start value  $a_0$  and the step value  $c$ . The basis we chose is convenient since it makes this fact explicit, as we multiply the first basis element by  $a_0$  and the second by  $c$  to obtain any arithmetic sequence of start value  $a_0$  and step value  $c$ . In fact, however, any two arithmetic sequences in  $W$  form a basis for that subspace as long as they are linearly independent; the proof is left as an exercise.

8. Let  $U$  and  $W$  be subspaces of a vector space  $V$ . We define two new subspaces as follows:

$$U + W = \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U, \mathbf{w} \in W\}$$

$$U \cap W = \{\mathbf{v} \in V \mid \mathbf{v} \in U \text{ and } \mathbf{v} \in W\}$$

- (a) (\*) Convince yourself that both  $U + W$  and  $U \cap W$  are, in fact, subspaces of  $V$ .  
(b) Show that if  $\dim(V) < \infty$ , then

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

- (a) The proof that  $U + W$  and  $U \cap W$  are subspaces of  $V$  involves showing closure.  
 (b) Let  $\mathcal{B}_{U \cap W} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  be a basis for  $U \cap W$ . Note that if  $U \cap W = \{\mathbf{0}\}$ , then  $\mathcal{B}_{U \cap W} = \emptyset$ . We can obtain bases for  $U$  and  $W$  by extending  $\mathcal{B}_{U \cap W}$ :

$$\begin{aligned}\mathcal{B}_U &= \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_n\} \\ \mathcal{B}_W &= \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}_1, \dots, \mathbf{w}_m\}\end{aligned}$$

Note that if  $U \cap W = U$ , no new basis elements are added in the extended basis for  $U$ , and similarly, if  $U \cap W = W$ , no new basis elements are added in the extended basis for  $W$ .

Now, consider the set  $\mathcal{B}_U \cup \mathcal{B}_W = \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{w}_1, \dots, \mathbf{w}_m\}$ . Any vector in  $U + W$  can be expressed as a linear combination of vectors in this set, and the set is linearly independent since the  $\mathbf{u}_i$ s and  $\mathbf{w}_i$ s cannot non-trivially combine to equal  $\mathbf{0}$  (otherwise they would be in  $U \cap W$  in the first place). This set is therefore a basis for  $U + W$ . Therefore we have:

$$\begin{aligned}\dim(U + W) &= k + n + m \\ &= (k + n) + (k + m) - k \\ &= \dim(U) + \dim(W) - \dim(U \cap W).\end{aligned}$$

Note that this solution works because we built bases for  $U$  and  $W$  from  $U \cap W$  up, not the other way around; it is very difficult in general to extract a basis for  $U \cap W$  from bases  $U$  and  $W$ .

There is an alternative solution that works even in the case that  $V$  is infinite-dimensional, which will make sense now that you have learned direct sums. Recall that the direct sum  $U \oplus W$  is the result of imposing a natural vector space structure on the cartesian product  $U \times W$ .

Define a linear map  $L : U \oplus W \rightarrow V$  such that  $L(\mathbf{u}, \mathbf{w}) = \mathbf{u} - \mathbf{w}$  for all  $\mathbf{u} \in U$  and  $\mathbf{w} \in W$ .

Then  $\text{Im}(L) = \{\mathbf{u} - \mathbf{w} \mid \mathbf{u} \in U, \mathbf{w} \in W\} = \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U, \mathbf{w} \in W\} = U + W$ . Also,  $\text{Ker}(L) = \{(\mathbf{u}, \mathbf{w}) \mid \mathbf{u} - \mathbf{w} = \mathbf{0}\}$ . Now,  $\mathbf{u} - \mathbf{w} = \mathbf{0}$  implies  $\mathbf{u} = \mathbf{w}$  and therefore  $u \in U \cap W$ ,  $w \in U \cap W$ . Hence  $\text{Ker}(L) = \{(\mathbf{v}, \mathbf{v}) \mid \mathbf{v} \in U \cap W\}$  and hence is isomorphic to  $U \cap W$ , with the isomorphism  $\varphi(\mathbf{v}, \mathbf{v}) = \mathbf{v}$ .

By the rank-nullity theorem,  $\dim(\text{Ker}(L)) + \dim(\text{Im}(L)) = \dim(U \oplus W)$ . Now  $\text{Ker}(L) \cong U \cap W$  and  $\text{Im}(L) = U + W$ , and it is left as an exercise to see that  $\dim(U \oplus W) = \dim(U) + \dim(W)$ . Therefore, we conclude that  $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$  as required.