Solution Set 3D

Daniel Gardiner

October 27, 2004

Math 23a
Prof. Boller

9 (D) Let $P_n(\mathbb{R}) = \{p(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R}, \forall i\}$ be the vector space of all polynomials of degree less than or equal to n. Consider the map $L : P_n(\mathbb{R}) \to \mathbb{R}$ defined by $L(p) = \int_0^1 p(x)\,dx$.

(a) Show that L is a linear map.

Solution: To show that L is linear, we must prove that $L(p+q) = L(p) + L(q)$ for any $p, q \in P_n$ and that $L(cp) = cL(p)$ for any $p \in P_n, c \in \mathbb{R}$. Noticing, then, that by properties of integration,

$$L(p+q) = \int_0^1 (p(x) + q(x))\,dx = \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx = L(p) + L(q),$$

and that

$$L(cp) = \int_0^1 cp(x)\,dx = c\int_0^1 p(x)\,dx = cL(p),$$

we have shown that L is indeed a linear map.

(b) Determine $Im(L)$, and find a basis.

Solution: We note that for any $a \in \mathbb{R}$, we can choose $p(x) = a$, such that

$$\int_0^1 p(x)\,dx = \int_0^1 a\,dx = a.$$

Hence, $Im(L) = \mathbb{R}$, and so our basis is simply $\{1\}$.

13 (D) Consider the linear differential operator $D : C^\infty \longrightarrow C^\infty$ given by $D(f) = f' + af$, where a is some fixed real number. (Recall that C^∞ is the vector space of all functions which are infinitely-differentiable.)

(a) Find $Ker(D)$

Solution: We note that $Ker(D) = \{f \in C^\infty | f' + af = 0\}$ where a is some fixed real number. In other words, we want to solve the differential equation $f' = -af$. Writing f' as dy/dx, we have:

$$dy/dx = -ay.$$

Rearranging terms, we get that $dy/y = -adx$. Thus, $lny = -ax + C$, for $C \in \mathbb{R}$. Hence:

$$y = Ce^{-ax}.$$

And so, $Ker(D) = \{f \in C^\infty | f(x) = Ce^{-ax}$, for $C \in \mathbb{R}\}$, where a is some real number fixed by our differential operator.

(b) Show that D is surjective.

Solution: We want to show that for any $g \in C^\infty$, there exists an $f \in C^\infty$ s.t. $f' + af = g$.

Multiplying by e^{ax}, we have:

$$f'e^{ax} + afe^{ax} = ge^{ax}.$$

But noticing that $f'e^{ax} + afe^{ax} = d(fe^{ax})/dx$ gives us:

$$fe^{ax} = \int ge^{ax}\, dx.$$

Since $g \in C^\infty$ by assumption, and we know that $e^{ax} \in C^\infty$, g and e^{ax} must be continuous. Hence, ge^{ax} is continuous as well, and we know that all continuous functions are integrable, so $\int ge^{ax}\, dx$ must exist. Additionally, as e^{ax} can never equal 0, we can multiply our equation by $1/e^{ax}$ to solve for f.

Hence, $f = \int ge^{ax}\, dx \; / \; e^{ax}$, and noticing that f is just the quotient of two infinitely differentiable functions, we can conclude that $f \in C^\infty$. Thus, for any $g \in C^\infty$, there exists an $f \in C^\infty$ s.t. $D(f) = g$.

Notes: While proofs were on the whole excellent (definitely the best I've seen so far), I wanted to note a few common mistakes:

1: Many people attempted to justify the existence of $\int ge^{ax}\, dx$ by claiming that $f \in C^\infty$. Remember that while $g \in C^\infty$ by assumption, we are trying to prove the existence of an $f \in C^\infty$ that satisfies our differential equation.

2: Another common error was the claim that g could not be a constant since $g \in C^\infty$. Constant functions are infinitely differentiable - their derivatives ($f = 0$) just are not very exciting.

3: Finally, many people attempted to justify the existence of $\int ge^{ax}\, dx$ by citing the Fundamental Theorem of Calculus. While the Fundamental Theorem contributes to just about every integration result, simply citing it is insufficient: explain your reasoning.