Math 23a Solution: Problem E

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(E) *In this problem, we consider the* shift *operator. Consider the linear map* $S : \mathbb{R}^3 \to \mathbb{R}^3$ *which acts as follows:*

$$S(x, y, z) = (0, x, y).$$

Find the kernel and image of S*, and verify that*

$$dim(Ker(S)) + dim(Im(S)) = dim(\mathbb{R}^3).$$

We have $(x, y, z) \in$ker S iff $S(x, y, z) = (0, 0, 0)$, or $(0, x, y) = (0, 0, 0)$ which means that $x = 0$ and $y = 0$. Conversely, we have $S(0, 0, z) = (0, 0, 0)$ for any $z \in \mathbb{R}$. Thus ker S =span(e_3), in the standard basis for $\mathbb{R}^3$, and so dim ker S=1. For the image, we have $S(x, y, z) = (0, x, y) \in$Span($e_2, e_3$). Conversely, for any $(0, x, y) \in$Span(e_2, e_3), we have $S(x, y, 0) = (0, x, y)$. Thus im S =span(e_2, e_3), so that it has dimension 2. Since dim $\mathbb{R}^3 = 3$, we have dim ker S+dim im S=1+2=3=dim $\mathbb{R}^3$, as desired.

(E) *We generalize the notion of the* shift *operator. Let* V *be the vector space of all infinite sequences of real numbers as in Problem #2.11 and # 3.5 above, and consider the linear maps* $S : V \to V$ *and* $T : V \to V$*, where* S *and* T *act as follows:*

$$S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)$$

$$T(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots)$$

(a) *Find the kernel and image of* S*. How does the result about the dimensions of kernels and images apply?*

We have $S(a_0, a_1, \ldots) = \mathbf{0}$ iff $(0, a_0, a_1, \ldots) = (0, 0, 0, \ldots)$. Equating the $i+$1'th entries gives $a_i = 0$ for all $i = 0, 1, 2 \ldots$. Thus the only element in the kernel of S is $\mathbf{0}$. As for the image, we note that everything in the image of S begins with a zero, and further that for any such sequence $(0, a_0, a_1, \ldots)$ we have $S(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots)$, and thus im S is all the sequences that start with a zero. It is clear that im S is infinite dimensional, as is the whole space R^∞. Thus the rank-nullity theorem, as we proved it, does not apply, since it assumes that the space is finite dimensional. You could, however, say that both im S and $\mathbb{R}^\infty$ have "dimension" ∞, and ker S has dimension 0, so the Rank-Nullity theorem amounts to the plausible looking $\infty + 0 = \infty$. In order to really make sense of this, however, we would have to generalize what we mean by dimension to the case of infinite dimensional spaces, and do a lot of work to ensure everything worked.

(b) *Show that $T \circ S = I$ but that $S \circ T \neq I$, where $I : V \longrightarrow V$ is the identity map.*

We have

$$T \circ S(a_0, a_1, \ldots) = T(S(a_0, a_1, \ldots) = T(0, a_0, a_1 \ldots) = (a_0, a_1, \ldots)$$

for any sequence $(a_0, a_1, \ldots)$, and thus $T \circ S = I$. For the other part, note that $T(1, 0, 0, \ldots) = (0, 0, 0, \ldots) = \mathbf{0}$. Thus $S \circ T(1, 0, 0 \ldots) = S(\mathbf{0}) = \mathbf{0} \neq (1, 0, 0 \ldots)$, so it cannot be the identity.

(c) *Which of S and T is onto? Which is one-to-one? Which is invertible? Explain.*

It is a starred problem on this problem set that a linear map is injective iff its kernel is trivial (i.e. the only thing in the kernel is $\mathbf{0}$). This was also mentioned in class, and since I am going to use this fact here, I may as well quickly prove it. Assume $T : V \to W$ is injective. Assume $T(\mathbf{v}) = \mathbf{0}$. Since T is linear, $T(\mathbf{0}) = \mathbf{0}$, and thus $T(\mathbf{v}) = T(\mathbf{0})$. Applying the definition of injectivity, we see that $\mathbf{v} = \mathbf{0}$, and so $\mathbf{0}$ is the whole kernel. Conversely, assume the kernel is trivial. Let $T(\mathbf{v}) = T(\mathbf{w})$. Subtracting $T(\mathbf{w})$ from both sides and using linearity, we have $T(\mathbf{v} - \mathbf{w}) = \mathbf{0}$. Thus $\mathbf{v} - \mathbf{w}$ is in the kernel of T. But the kernel of T is trivial, so $\mathbf{v} - \mathbf{w} = \mathbf{0}$, or $\mathbf{v} = \mathbf{w}$. Thus T is injective.

We now return to the problem at hand. We showed in part a) that S has a trivial kernel, and thus is injective. In part b), we saw that $T(1, 0, 0 \ldots) = \mathbf{0}$, so T has non-zero vectors in its kernel, and cannot be injective.

We noted above that everything in im S has a 0 as its first entry, and so $(1, 0, 0 \ldots)$ cannot be in the image of S, and thus S is not surjective. T, however, is surjective: for any $\mathbf{x} \in \mathbb{R}^\infty$, we have $T(S(\mathbf{x})) = \mathbf{x}$, since $T \circ S$ is the identity. Thus x is in the image of T, and T is surjective.

To be invertible, a function must be injective and surjective. Since neither T nor S is injective and surjective, neither is invertible. Actually, it is a fact (which I won't prove) that for any two functions A and B (not necessarily linear) which map some set X to itself, if $A \circ B$ is the identity but $B \circ A$ is not the identity, then neither A nor B is invertible. Using this general result, we could immediately conclude from b) that neither T nor S is invertible.