Solution Set 4B

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October 27, 2004

Math 23a
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9 (D) Let F be any field. Show that the linear operator $L : F^2 \to F^2$ given by $L(x, y) = (ax + by, cx + dy)$ is invertible if and only if $ad - bc \neq 0$.

(a) **Solution:** To simplify the proof, I first claim the following fact:

Fact: If a linear operator on a finite dimensional vector space is injective, it is surjective and hence bijective as well.

Proof: Let $L : V \to V$, where V is some finite dimensional vector space. If L is injective, then $Ker(L)$ is trivial, and in particular, $dim(Ker(L)) = 0$. Applying Rank-Nullity, we have, then, that $dim(V) = dim(Im(L)) + dim(Ker(L)) = dim(Im(L)) + 0$. Hence, $dim(V) = dim(Im(L))$; but L was assumed to be a linear operator, so $Im(L)$ is a subspace of V. Thus, $Im(L) = V$, as any basis for $Im(L)$ must also be a basis for V, so L is surjective and therefore bijective as well.

Now let's get back to the problem at hand. Noting that the above fact certainly applies for our linear operator, we consider $Ker(L)$.
Clearly, $(x, y) \in Ker(L) => ax + by = 0, cx + dy = 0$ $(eqs.1, 2)$.

Multiplying $(eq.1)$ by c and $(eq.2)$ by a, we have $cax + bcy = 0$ and $cax + ady = 0$ $(eqs.3, 4)$, and subtracting (3) from (4), we get $ady - bcy = 0$. Factoring out y, we are left with $(ad - bc)y = 0$ $(eq.5)$.

Similarly, multiplying equation 1 by d and equation 2 by b, we get
$adx + bdy = 0, bcx + bdy = 0$, and subtracting the second of these equations from the first as above, we have $adx - bcx = 0 => (ad - bc)x = 0$ $(eq.6)$.

Now, if $ad - bc \neq 0$, equations 5 and 6 tell us that $x = y = 0$. Hence, L is injective and hence bijective and invertible by our fact. In the other direction, note that if $ad - bc = 0$, then any (x, y) satisfies equations 5 and 6. Hence, if $ad - bc = 0$, our function can not possibly be invertible since it is not even injective. Thus, we can conclude that invertibility implies that $ad - bc \neq 0$ as well.

Notes: On the whole, there were a number of common errors or otherwise dubious moves, so I have a number of notes.

1: Since in this question we are working in an abstract field, half of the battle is actually solving our simultaneous equations given by considering $(x, y) \in Ker(L)$. Hence, not showing work was penalized. More to the point, many people attemped at some point to divide by constants that could be 0; for example, many proofs required dividing by a. Remember that multiplicative inverses only exist in a field for non-zero elements. It's certainly possible to have $a = 0$ and $ad - bc \neq 0$, so be careful.

2. Also of note, there were a number of dubious proofs by contradiction. Be sure to check the overall logic of your work. Some proofs, for example, began by considering the case where $ad-bc = 0$. Showing that $ad-bc = 0$ implies that L is not invertible does **not** show that $ad-bc \neq 0$ implies that L **is** invertible - it only shows the other direction (namely that L invertible $=>$ $ad - bc \neq 0$.) Similarly, beginning by considering the case where $a = b = c = d = 0$ and deducing that L is not invertible tells you almost nothing at all.

3. Now that we're on the fourth problem set, coherence is important. This class is about writing proofs, and everyone has been reminded repeatedly to use complete sentences, have a clear setup to their solution, etc. Handing in sheets of equations with no explanation is unacceptable; it was punished slightly on this problem set (maybe about half a point), and will be penalized more severely (by me at least) hereafter. Be sure to hand in something neat, legible, and clear. State your assumptions, show all your work, and explain every step.

4. As an aside, many proofs relied on explicitly constructing L inverse. In showing that $ad - bc \neq 0 => L$ is invertible, this is certainly fine, but the opposite direction is not preferable. If you really want to make the argument that L invertible implies $ad - bc \neq 0$ through looking at the inverse function, you need to argue that your inverse function is the only possible one; in other words, that L inverse could not look any different. Then, I suppose, you can say that if $ad - bc = 0$, L inverse is not defined, but this is still just a bit indirect. In my opinion, the clearest proofs proceeded by analyzing the bijectivity of L instead.

5. Similarly, many people went one step farther than my above proof, and divided

through by $ad-bc$, then concluded by saying something like: if $ad-bc$=0, then our equation is undefined." While this is true, the whole point is that if $ad - bc = 0$, its multiplicative inverse doesn't exist in the first place, and you should not be dividing by it at all. In other words, be very specific with what you write. Note that I did not take off for this, however.

6. Finally, I chose to prove the injectivity fact used in this proof in general because I think that it's quite useful. Anyone looking for more practice should prove to themselves that for a linear operator in a finite dimensional vector space, surjectivity implies injectivity and bijectivity as well.