SOLUTION SET 4C

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Math 23a
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8. Let V be a vector space. We say that the linear map $P : V \to V$ is a projection provided that $P^2 = P$, that is: $P(P(\mathbf{v})) = P(\mathbf{v}), \forall \mathbf{v} \in V$. Define two subspaces of V as follows:

$$V_0 = \{\mathbf{v} \in V \mid P(\mathbf{v}) = \mathbf{0}\}$$

$$V_1 = \{\mathbf{v} \in V \mid P(\mathbf{v}) = \mathbf{v}\}$$

Show that $V = V_0 \oplus V_1$.

We shall prove that $V_0 \cap V_1 = \{\mathbf{0}\}$ and $V = V_0 + V_1$.

First, if $\mathbf{v}$ is a vector in $V_0 \cap V_1$, then $P(\mathbf{v}) = \mathbf{0}$ since $\mathbf{v} \in V_0$ and $P(\mathbf{v}) = \mathbf{v}$ since $\mathbf{v} \in V_1$. Therefore $\mathbf{v} = \mathbf{0}$, and $V_0 \cap V_1$ is just the trivial subspace $\{\mathbf{0}\}$.

Second, let $\mathbf{v}$ be a vector in V. We can write:

$$\mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v} + (-P(\mathbf{v}) + P(\mathbf{v})) = (\mathbf{v} - P(\mathbf{v})) + P(\mathbf{v})$$

Now, $P(\mathbf{v} - P(\mathbf{v})) = P(\mathbf{v}) - P(P(\mathbf{v})) = \mathbf{0}$, so $\mathbf{v} - P(\mathbf{v})$ is a vector in V_0. Also, as $P(P(\mathbf{v})) = P(\mathbf{v})$, $P(\mathbf{v})$ is a vector in V_1. This proves that any $\mathbf{v} \in V$ can be written as the sum of a vector in V_0 and a vector in V_1, that is, $V = V_0 + V_1$.

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