# Solution Set 5A

## Daniel Gardiner

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Math 23a
Prof. Boller

2 (A) Suppose $\lambda$ is an eigenvalue for the linear transformation $A : V \to V$.

(a) Show that $\lambda^2$ is an eigenvalue for $A^2$.

**Solution:** If $\lambda$ is an eigenvalue for A, then $\exists$ a non-zero $v \in V$ s.t. $A(v) = \lambda v$. I claim that this $v$ is also an eigenvector for $A^2$, with eigenvalue $\lambda^2$, as $A^2(v) = A(A(v)) = A(\lambda v) = \lambda(A(v)) = \lambda * \lambda v = \lambda^2 v$. (Notice that we have used the fact that $\lambda$ is a scalar to pull it outside of the linear transformation, $A$). Hence, $v$ is a non-zero eigenvector with eigenvalue $\lambda^2$, as desired.

(b) If $A$ is invertible, show that $\lambda^{-1}$ is an eigenvalue for $A^{-1}$. (What happens if $\lambda = 0$?)

**Solution:** Since $A$ is invertible, it is bijective, and in particular injective. Hence $Ker(A)$ is trivial, so we can not have $\lambda = 0$.

With that in mind, we consider our non-zero $v \in V$ s.t. $A(v) = \lambda v$. As $A$ is invertible, we can consider $A^{-1}(A(v))$. Clearly, $A^{-1}A = I$, so on the one hand, we have that $A^{-1}(A(v)) = v$, but we also know that $A(v) = \lambda v$. Hence, $v = A^{-1}(A(v)) = A^{-1}(\lambda v) = \lambda A^{-1}(v)$, and as $\lambda \neq 0$, as shown above, we can multiply through by $\lambda^{-1}$, obtaining $A^{-1}(v) = \lambda^{-1} v$. Thus, $\lambda^{-1}$ is an eigenvalue for $A^{-1}$.

**Notes:** Not a whole lot to say here: just note that it is incorrect to state that $\lambda \neq 0$ only because it is an eigenvalue. There certainly are transformations with eigenspaces corresponding to $\lambda = 0$ - these operators are just not invertible. Additionally, remember to take a non-zero eigenvector, as the zero vector is always an eigenvector for any eigenvalue; in order to be an eigenvalue, $\lambda$ must correspond to a non-trivial eigenspace - those of you who are observant might have noticed that I myself forgot this in the first version of this proof.

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