Math 23a Solution: Problem B

David Jackson-Hanen

November 18, 2004

We proceed by induction on m in the problem statement. The case $m = 1$ is obvious, since eigenvectors are non-zero and a single non-zero vector is linearly independent. Now assume the theorem holds for any collection of m eigenvectors with distinct eigenvalues, and let $\{v_1 \dots v_{m+1}\}$ be eigenvectors of A with distinct eigenvalues $\lambda_1, \dots \lambda_{m+1}$. Assume

$$(*) \qquad a_1 v_1 + \cdots a_{m+1} v_{m+1} = 0.$$

Applying A to $(*)$ and using linearity gives

$$a_1 \lambda_1 v_1 + \cdots a_{m+1} \lambda_{m+1} v_{m+1} = 0$$

while multiplying $(*)$ by λ_1 gives

$$a_1 \lambda_1 v_1 + \cdots a_{m+1} \lambda_1 v_{m+1} = 0.$$

Subtracting these equations gives

$$a_2(\lambda_2 - \lambda_1) + \cdots + a_{m+1}(\lambda_{m+1} - \lambda_1) v_{m+1} = 0.$$

By our inductive hypothesis $\{v_2 \dots v_{m+1}\}$ is linearly independent, so all the coefficients above must be zero. But $\lambda_i - \lambda_1 \neq 0$ for $2 \leq i \leq m+1$, since the eigenvalues are distinct, and so $a_i = 0$. Going back to $(*)$ gives $a_1 v_1 = 0$, or $a_1 = 0$ as well. Thus $\{v_1 \dots v_{m+1}\}$ is linearly independent, and this completes the induction.