

## SOLUTION SET 5C

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5. Two linear transformations  $A : V \rightarrow V$  and  $B : V \rightarrow V$  are said to be *similar* if there exists an invertible linear transformation  $S : V \rightarrow V$  such that  $A = SBS^{-1}$ . Consider the following, where a well-written answer may suffice for both parts.

- (a) Show that if  $A$  and  $B$  are similar, then  $\text{Spec}(A) = \text{Spec}(B)$

**Solution:** Let  $\lambda \in \text{Spec}(A)$ . Then, there exists  $v \in V$ ,  $v \neq 0$  such that  $Av = \lambda v$ . Then, because  $A = SBS^{-1}$ , we have that  $SBS^{-1}v = \lambda v$ . Multiplying by  $S^{-1}$  on both sides, and invoking the linearity of  $S^{-1}$  gives us  $BS^{-1}v = \lambda S^{-1}v$ . To see that  $\lambda$  is an eigenvalue of  $B$  (with eigenvector  $S^{-1}v$ ) we need only check that  $S^{-1}v \neq 0$  (0 is by definition not an eigenvector). This follows from the fact that both  $S$  and  $S^{-1}$  are invertible, and thus have trivial kernel; in particular, since  $v \neq 0$ ,  $S^{-1}v \neq 0$ . Hence, we have shown that  $\text{Spec}(A) \subset \text{Spec}(B)$ .

Similarly, for the reverse inclusion, if  $\mu$  is an eigenvalue of  $B$  with eigenvector  $w$ , then we must have that  $Bw = \mu w$ . Then, since  $B = S^{-1}AS$ , it follows that  $S^{-1}ASw = \mu w$ , so that  $ASw = \mu Sw$  (we have multiplied both sides by  $S$  and used linearity). By the same argument as above,  $Sw \neq 0$ , so that  $\mu$  is an eigenvalue of  $A$  with eigenvector  $Sw$ . Hence,  $\text{Spec}(B) \subset \text{Spec}(A)$ , and together with the first part, we get that  $\text{Spec}(A) = \text{Spec}(B)$ , as desired.

- (b) Suppose  $A$  and  $B$  are similar and  $\lambda$  is an eigenvalue of both. Find the precise relationship between the eigenspace for  $\lambda$  with respect to  $A$  and the eigenspace for  $\lambda$  with respect to  $B$

**Solution:** As we saw in part (a), if  $\lambda$  is an eigenvalue of  $A$  with eigenvector  $v$ , then it is an eigenvalue of  $B$  with eigenvector  $S^{-1}v$ . Hence, if we let  $V_\lambda^A$  be the eigenspace for  $\lambda$  with respect to  $A$  and  $V_\lambda^B$  represent the eigenspace for  $\lambda$  with respect to  $B$ , we find that  $V_\lambda^B = S^{-1}(V_\lambda^A)$ .

Most people did very well on this problem. The only issue was that a vast majority forgot to check that their putative eigenvectors were nonzero. Also, a few people proved that  $\text{Spec}(A) \subset \text{Spec}(B)$ , without proving the reverse inclusion; a simple sentence saying that the proof is exactly the same would have been enough.