

# Math 23a Solution: Problem C

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November December 10, 2004

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We will actually prove something stronger than the problem asks, namely, that if  $f : V^k \rightarrow F$  is any multilinear map (not necessarily alternating) for  $k < n = \dim V$ , then there is a set of linearly independent vectors  $\{w_1, \dots, w_k\}$  such that  $f(w_1, \dots, w_k) = 0$  (proof due to Luke Li). Let  $\{v_1, \dots, v_n\}$  be a basis, and let  $r = f(v_1, \dots, v_{k-1}, v_k)$ ,  $s = f(v_1, \dots, v_{k-1}, v_{k+1})$ . If either  $r = 0$  or  $s = 0$ , then we are done, because the sets  $\{v_1, \dots, v_{k-1}, v_k\}$  and  $\{v_1, \dots, v_{k-1}, v_{k+1}\}$  are linearly independent. If  $r \neq 0$  and  $s \neq 0$ , then the set  $\{v_1, \dots, v_{k-1}, sv_k - rv_{k+1}\}$  is linearly independent, and

$$\begin{aligned} f(v_1, \dots, v_{k-1}, sv_k - rv_{k+1}) &= sf(v_1, \dots, v_k) - rf(v_1, \dots, v_{k+1}) & (1) \\ &= sr - rs & (2) \\ &= 0 & (3) \end{aligned}$$

This is undoubtedly the slickest way to do this, but the problem statement strictly speaking asked for a proof by example, so to be a little more concrete I will also give an example, adapted from many of your solutions. Define  $P : V \rightarrow F^k$  by  $P(a_1v_1 + \dots + a_nv_n) = a_1e_1 + \dots + a_ke_k$ . Clearly  $P$  is linear. Let  $d$  be the unique  $k$ -form on  $F^k$  with  $d(e_1, \dots, e_k) = 1$ . Define  $f(w_1, \dots, w_k) := d(Pw_1, \dots, Pw_k)$ . Since  $P$  is linear and  $d$  is an alternating form,  $f$  is an alternating form. It is non-trivial, since  $f(v_1, \dots, v_k) = d(e_1, \dots, e_k) = 1$ , but  $f(v_2, \dots, v_k, v_n) = d(e_2, \dots, e_n, 0) = 0$ , so  $f$  vanishes on a set of linearly independent vectors.

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(b)  $f$  is simply the determinant of the two-by-two matrix with columns equal to the two vectors  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , which we already know is alternating and multilinear in the columns. It is also simple to show directly that  $f$  is multilinear; for instance,  $f(k(a, b) + l(a', b'), (c, d)) = (ka + la')d + (kb + lb')c = k(ad - bc) + l(a'd - bc') = kf((a, b), (c, d)) + lf((a', b'), (c, d))$ , and the proof of linearity in the second entry is almost identical. It is alternating because  $ab - ba = 0$ .

(c) In the next part we will construct a basis for the 2-forms on  $\mathbb{R}^3$ ; any two elements of the basis are independent.

(d) We proved in class that the space of  $n$ -forms on an  $n$  dimensional space is one dimensional, so the dimension of the space of alternating bilinear forms on  $\mathbb{R}^2$  is 1.

As for  $\mathbb{R}^3$ , let  $f$  be an alternating bilinear form, and let  $v = a_1e_1 + a_2e_2 + a_3e_3$ ,  $w = b_1e_1 + b_2e_2 + b_3e_3$  be any two vectors. Then using bilinearity and skew-symmetry, and a bit of algebra, we have

$$f(v, w) = (a_1b_2 - a_2b_1)f(e_1, e_2) + (a_1b_3 - a_3b_1)f(e_1, e_3) + (a_2b_3 - a_3b_2)f(e_2, e_3)$$

Thus  $f$  is completely determined by the three values  $f(e_1, e_2)$ ,  $f(e_2, e_3)$  and  $f(e_1, e_3)$ , so we expect the dimension to be three. Let  $f_1(a_1e_1 + a_2e_2 + a_3e_3, b_1e_1 + b_2e_2 + b_3e_3) = a_1b_2 - a_2b_1$ ,  $f_2(a_1e_1 + a_2e_2 + a_3e_3, b_1e_1 + b_2e_2 + b_3e_3) = a_1b_3 - a_3b_1$ , and  $f_3(a_1e_1 + a_2e_2 + a_3e_3, b_1e_1 + b_2e_2 + b_3e_3) = a_2b_3 - a_3b_2$ . Arguments very similar to those in part b) show that these are all bilinear alternating forms. Note that  $f_1(e_1, e_2) = 1$ , while  $f_1(e_1, e_3) = f_1(e_2, e_3) = 0$ . Similarly,  $f_2$  and  $f_3$  are non-zero on  $(e_1, e_3)$  and  $(e_2, e_3)$ , respectively. Thus if  $c_1f_1 + c_2f_2 + c_3f_3 = 0$ , (note that the zero on the right is the zero form), operating on  $(e_1, e_2)$  gives  $c_1 = 0$ . Similar arguments show  $c_2 = c_3 = 0$ , and so these forms are independent. On the other hand, our calculations above show that if  $f$  is any bilinear form, then  $f(v, w) = f(e_1, e_2)f_1(v, w) + f(e_1, e_3)f_2(v, w) + f(e_2, e_3)f_3(v, w)$ , or  $f = f(e_1, e_2)f_1 + f(e_1, e_3)f_2 + f(e_2, e_3)f_3$ , so these forms span. They thus form a basis, and so the dimension is three.