Solution Set 7C

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(C) Consider the real numbers as defined by equivalence classes of Cauchy sequences of rational numbers.

(8) Name the equivalence class that acts as the multiplicative identity, and verify that it does.

Solution: I claim that $[\{1\}_{n=1}^{\infty}]$ is the multiplicative identity. Note first of all that $\{1\}_{n=1}^{\infty}$ is Cauchy, as given any $\epsilon > 0$, we can take $N = 1$, noting that $\forall i, j > N$, $|a_i - a_j| = |1 - 1| = 0 < \epsilon$. Moreover, $[\{1\}_{n=1}^{\infty}]$ is the multiplicative identity, as given any $[\{a_n\}_{n=1}^{\infty}]$, $[\{a_n\}_{n=1}^{\infty}]*[\{1\}_{n=1}^{\infty}]=[\{a_n * 1\}_{n=1}^{\infty}]=[\{a_n\}_{n=1}^{\infty}]$. Notice that we have applied our definition for multiplication of equivalence classes of Cauchy sequences of rationals.

(9) Prove the existence of multiplicative inverses (for elements other than the additive identity).

Solution: Let $[\{a_n\}_{n=1}^{\infty}]$ be some element that is not the additive identity. In other words, $[\{a_n\}_{n=1}^{\infty}] \neq [\{0\}_{n=1}^{\infty}]$. I first claim the following lemma.

Lemma: Let $[\{a_n\}_{n=1}^{\infty}]$ be some element other than the additive identity. Then $\exists N \in \mathbb{N}, L > 0$ s.t. $|a_n| \geq L \forall n > N$, for some $L \in \mathbb{Q}$.

Proof: By contradiction. Take $\epsilon > 0$. Since $\{a_n\}_{n=1}^{\infty}$ is Cauchy, $\exists N_1 \in \mathbb{N}$ s.t. $|a_n - a_m| < \epsilon/2 \ \forall \ n, m > N_1$. Now assume that there were no such L, in other words, that past N_1 our sequence was not bounded below by a non-zero L. Then for any $\epsilon > 0$, we can find some $x > N_1$ s.t. $|a_x| < \epsilon$. In particular, then, for $\epsilon' = \epsilon/2, \exists \ x > N$ s.t. $|a_x| < \epsilon'$. Applying our above Cauchy condition, we have then that $|a_n - a_x| < \epsilon/2 \ \forall n > N_1$. But this is equivalent to the fact that $-\epsilon/2 < a_n - a_x < \epsilon/2$ $(Eq.1)$, and since $|a_x| < \epsilon' = \epsilon/2$, and hence $-\epsilon/2 < a_x < \epsilon/2$ $(Eq.2)$, we can add equations 1 and 2 to conclude that $-\epsilon < a_n < \epsilon, \forall n > N_1$.

But this is just the fact that $|a_n| < \epsilon \; \forall \; n > N_1$, which certainly implies that $|a_n - 0| < \epsilon \; \forall \epsilon > 0$, or in other words, that $\{a_n\}_{n=1}^{\infty}$ is equivalent to $\{0\}_{n=1}^{\infty}$. But this is a contradiction, as $[\{a_n\}_{n=1}^{\infty}]$ was assumed not to be the additive identity. Hence, past N_1, $\{a_n\}_{n=1}^{\infty}$ is bounded below by some non-zero L.

With that in mind, we consider our general $[\{a_n\}_{n=1}^{\infty}] \neq [\{0\}_{n=1}^{\infty}]$. Note that the above lemma implies that $\{a_n\}_{n=1}^{\infty}$ has only a finite number of 0s, as we can find an N_1 s.t. all terms past N_1 are bounded below by some non-zero L. Hence, define $\{b_n\}_{n=1}^{\infty}$, where $b_n = \begin{cases} a_n & \text{if } a_n \neq 0 \\ 1 & \text{if } a_n = 0. \end{cases}$ Notice that $\forall n > N_1$, $b_n = a_n$. Hence, $\{b_n\}_{n=1}^{\infty}$ is Cauchy, since, as $\{a_n\}_{n=1}^{\infty}$ is Cauchy, given any $\epsilon > 0$, $\exists N_2$ s.t. $|a_n - a_m| < \epsilon \; \forall \; n, m > N_2$. Taking N_{max} =max$\{N_1, N_2\}$, we have $|b_n - b_m| = |a_n - a_m| < \epsilon \; \forall n, m > N_{max}$. Likewise, $\{b_n\}_{n=1}^{\infty}$ is equivalent to $\{a_n\}_{n=1}^{\infty}$, since given any $\epsilon > 0$, $|a_n - b_n| = |a_n - a_n| = 0 < \epsilon, \forall n > N_1$.

Since $\{b_n\}_{n=1}^{\infty}$ has no zero terms, we thus define $[\{a_n\}_{n=1}^{\infty}]^{-1}$ $=[\{1/b_n\}_{n=1}^{\infty}]$. As $[\{b_n\}_{n=1}^{\infty}]$ is equivalent to $[\{a_n\}_{n=1}^{\infty}$, and thus is not equivalent to the additive identity, our above lemma certainly applies. Hence, $\exists \; M_1$ s.t. $|b_n| \geq L \; \forall n > M_1$, for some rational $L > 0$. Notice that this implies that $|1/b_n| \leq (1/L^2) \; \forall n > M_1$, as we have bounded the non-negative $|b_n|$ below by L. I claim, then, that $[\{1/b_n\}_{n=1}^{\infty}]$ is Cauchy, since, as $[\{b_n\}_{n=1}^{\infty}]$ is Cauchy, for any $\epsilon > 0$, we can take M_2 s.t. $|b_n - b_m| < (L^2\epsilon)$. Thus, $\forall n, m > M_{max}$ =max$\{M_1, M_2\}$, $|(1/b_n) - (1/b_m)| = |(b_n - b_m)/b_n b_m| = |b_n - b_m| * |1/b_n b_m| \leq |b_n - b_m| * (1/L^2) < (L^2\epsilon) * (1/L^2) = \epsilon$.. Hence, $[\{1/b_n\}_{n=1}^{\infty}]$ is Cauchy.

Since multiplication is well-defined for real numbers and $[\{a_n\}_{n=1}^{\infty}] = [\{b_n\}_{n=1}^{\infty}]$, $[\{a_n\}_{n=1}^{\infty}]$*$[\{1/b_n\}_{n=1}^{\infty}]$=$[\{b_n\}_{n=1}^{\infty}]$*$[\{1/b_n\}_{n=1}^{\infty}]$=$[\{(b_n * (1/b_n)\}_{n=1}^{\infty}]$=$[\{1\}_{n=1}^{\infty}]$, the multiplicative identity. Hence, we have constructed a multiplicative inverse for any $[\{a_n\}_{n=1}^{\infty}] \neq [\{0\}_{n=1}^{\infty}]$.

Notes: This one was really tough. Granted I was quite picky, but, after scaling, the highest score was a 19/20. I do think, however, that it's a really good question, as it forces you to understand the ins and outs of the construction of the reals. Thus, I certainly recommend reading over the above solution when studying Cauchy sequences, as this problem addresses many of the important facts that you will need to know for the exam. With that in mind, I have the following notes.

1. Obviously, proving that $[\{1/b_n\}_{n=1}^{\infty}]$ was Cauchy was the hardest part of the question. A number of people made a good effort to show this, but there were a few arguments that I was ultimately forced to penalize. First of all, many people recognized the need to bound $|b_n|$ in order to bound $|1/b_n|$ above. The first common mistake, however, was trying to bound $|b_n|$ above - this is possible, but doesn't get you anywhere in bounding $|1/b_n|$ above, as $(1/b_n)$ increases as b_n gets smaller. Others avoiding this pitfall suc-

ceeded in bounding $|1/b_n|$ below, but didn't show that there bound was $non - zero$. Remember that *any* absolute value is bounded below by zero - the challenge is showing that your sequence actually is bounded below past a certain N by a non-zero L.

2. Along these lines, a good check for any proof is to confirm that you have used all the given facts. Many people tried to prove the above lemma without actually using the fact that $[\{a_n\}_{n=1}^{\infty}] \neq [\{0\}_{n=1}^{\infty}]$. If you don't use this fact, your proof can not possibly work. Some tried to argue that since $\{a_n\}_{n=1}^{\infty}$ had a finite number of zeros, it had to be bounded below by a non-zero limit, but this is simply not true. Consider, for example, $[\{1/n\}_{n=1}^{\infty}]$ - this sequence converges to 0 without ever having a single zero term. The hard thing about this question is that you have to use both the fact that $\{a_n\}_{n=1}^{\infty}$ is Cauchy and the fact that it's not equivalent to the additive identity.

3. Another argument that I considered engaging but ultimately incorrect involved analyzing the limit of $\{a_n\}_{n=1}^{\infty}$. Concluding that as $\{a_n\}_{n=1}^{\infty}$ was not equivalent to $\{0\}_{n=1}^{\infty}$ and hence $\lim\limits_{n_1 \to \infty} a_n \neq 0$, many people proceeded by arguing that $\{a_n\}_{n=1}^{\infty}$ had to converge to some non-zero limit. Applying the defintion of a limit, they then bounded $|a_n|$ within, for example, $L - \epsilon$ for an appropriate ϵ in order to prove the above lemma. This argument, however, at least in my opinion, is problematic because we have not yet even *constructed* the reals. In particular, then, while it is certainly true that $\lim\limits_{n_1 \to \infty} a_n \neq 0$, this limit need not even exist - it could, for example, be π, or any irrational number for that matter. In other words, we don't yet have real numbers, so we have no notion of what convergence of a sequence around an irrational even means. Just as an aside on this point, remember that our Cauchy definition requires rational ϵ, so even had people convinced me that $|a_n|$ was bounded below by a non-zero irrational number, you still need to make some argument about the density of the rationals in the reals to really have a rigorous proof.

4. While the above three notes all refer to what were actually quite excellent proofs, the third one leads to a much deeper problem. It's very important to keep straight how this construction of the reals actually works. Remember that the real numbers actually *are* the equivalence classes of Cauchy sequences and **not** their limits. For the multiplicative identity question, for example, many people tried to argue that if you have a real number that converges to some number c, and you multiply it by something that converges to 1, the whole thing converges to c. While this is intuitively what is going on, it's not formally correct: the whole point is that we have to define the real numbers in terms of what we have, the rationals, rather than their limits, which might be irrational.

5. The final "deep" problem involved incorrect applications of the Cauchy condition. Remember that your ϵ must be a constant. Thus, trying to define an $\epsilon' = \epsilon / a_n a_m$

was problematic because a_n and a_m are variables. Similarly, trying to bound a_n by a_m does not help because a_n and a_m are not fixed. The point, then, is that to apply the Cauchy condition, you need to fix your ϵ.

6. Finally, a number of proofs tried to make $a_n > a_m$, claiming that this did not distort generality, but this does not really even make sense since there's no relationship between n and m - they're basically interchangeable, so this could not possibly be true for all n and m.