Math 23a Solution: Problem D

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(a) We note that by problem 6 if a and b are positive then $a \leq b$ iff $a^2 \leq b^2$; this fact will be used liberally and without comment. Proceeding by induction, we have $a_1 = 1 < 2$. If $a_n < 2$, then $a_n + 1 < 3$, so $a_{n+1} = \sqrt{a_n + 1} < \sqrt{3} < 2$. Thus $a_n < 2$ for all $n \in \mathbb{N}$.
(b) Again proceeding by induction, $a_1 = 1 < \sqrt{2} = \sqrt{1+1} = a_2$. If $a_n < a_{n+1}$, then $a_n + 1 < a_{n+1} + 1$, and $a_{n+1} = \sqrt{a_n + 1} < \sqrt{a_{n+1} + 1} = a_{n+2}$. Thus $a_n < a_{n+1}$ for all n.
(c) (a_n) form a bounded monotonically incresing sequence of real numbers, and hence converge to some number L by the completeness axiom for the real numbers. Since $\lim_{n\to\infty} a_n = L$, we have $\lim_{n\to\infty} a_{n+1} = L$, and so $\lim_{n\to\infty} a_{n+1}^2 = (\lim_{n\to\infty} a_{n+1})^2 = L^2$. By definition, $a_{n+1}^2 = a_n + 1$, so $L^2 = \lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty} a_n + 1 = L + 1$, so $L^2 - L - 1 = 0$.
(d) Using the quadratic formula, we see that the only possibilities for L are $(1 \pm \sqrt{5})/2$. However, $a_1 = 1$ and the sequence is increasing, and so $L > 1$, and cannot equal the negative solution. Hence we must have $L = \frac{1+\sqrt{5}}{2} = \varphi$ as desired. By the way, not justifying ruling out the negative solution was responsible for most of the points lost on this problem.