SOLUTION SET 7E

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12. In this problem, you will show that every real number (with a few exceptions) has a unique decimal expansion.

Let $x > 0$ be a a real number. Show that there is an integer k and integers $a_i \in \{0, 1, 2, \ldots, 9\}$ for every $i \geq k$ such that x may be represented in the form:

$$(1)\quad x \;=\; \sum_{i=k}^{\infty} a_i \cdot 10^{-i}$$

$$(2)\quad =\; a_k \cdot 10^{-k} + a_{k+1} \cdot 10^{-k-1} + \ldots + a_{-1} \cdot 10^1 + a_0 \cdot 10^0 + a_1 \cdot 10^{-1} + a_2 \cdot 10^{-2} + \ldots$$

Consider the following set:

$$(3)\qquad\qquad B = \{b \in \mathbb{N} \cup \{0\} | 10^{b+1}\}$$

Now, B is nonempty since $\mathbb{N}$ is unbounded in $\mathbb{R}$ and thus so is B; being a nonempty subset of $\mathbb{N} \cup \{0\}$, B has a smallest element by the Well-Ordering Principle. Let b_0 be this element, and set $k = -b_0$.

Now, let us look at the set

$$(4)\qquad\qquad C_k = \{c \in \mathbb{N} \cup \{0\} | (c+1) \cdot 10^{-k} > x\}$$

Again because $\mathbb{N}$ is unbounded in $\mathbb{R}$, C_k is nonempty, and it has a smallest element a_k. We claim that $a_k \leq 9$. For, if $a_k \geq 10$, $a_k \cdot 10^{-k} \geq 10 \cdot 10^{-k}$, which is equal to $10^{-k+1} = 10^{b_0+1}$. Now, 10^{b_0+1} is greater than x, as $b_0 \in B$. What this means is that we've found a member of C_k smaller than a_k, namely $a_k - 1$, as $((a_k - 1) + 1) \cdot 10^{-k} > x$, so we have arrived at a contradiction. Thus, $a_k \leq 9$, i.e. $a_k \in \{0, 1, 2, \ldots, 9\}$.

Remark 1. *Technically, the Well-Ordering Principle only works when we are dealing with the natural numbers, but it is clear that we can extend it to, say, $\mathbb{N} \cup \{0\}$, which is what we have tacitly done here.*

Let $r_k = x - a_k \cdot 10^{-k}$. We claim that $r_k < 10^{-k}$. This follows immediately from the construction of C_k and our choice of $a_k \in C_k$ in particular the inequality $(a_k + 1) \cdot 10^{-k} > x$.

We proceed similarly. Let

$$(5)\qquad\qquad C_{k+1} = \{c \in \mathbb{N} \cup \{0\} | (c+1) \cdot 10^{-(k+1)} > r_k\}$$

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Again invoking the well-ordering principle we choose a_{k+1}, the smallest element of C_{k+1}, and argue that $a_{k+1} \leq 9$. (After all, if $a_{k+1} \geq 10$, then $a_{k+1} \cdot 10^{-(k+1)} \geq 10^{-k} > r_k$, so that $a_{k+1} - 1$ is a smaller element of C_{k+1}, a contradiction.) We let

(6)
$$r_{k+1} = r_k - a_{k+1} \cdot 10^{-(k+1)}$$

(7)
$$= x - (a_k \cdot 10^{-k} + a_{k+1} \cdot 10^{-(k+1)})$$

and establish that $r_{k+1} < 10^{-(k+1)}$ exactly as before.

In general, continuing the process, suppose we have found a_i for $i = k$ to m; let

(8)
$$y_m^{(k)} = \sum_{i=k}^{m} a_i \cdot 10^{-i}$$

and

(9)
$$r_m = x - y_m^{(k)}$$

where $r_m < 10^{-m}$ (inductive hypothesis). Defining C_{m+1} just as we have defined C_k and C_{k+1} in the first steps, we are able to find a_{m+1} and argue that $a_{m+1} \in \{0, 1, \ldots, 9\}$. Furthermore, we can establish that $r_{m+1} < 10^{-(m+1)}$ by exactly the same methods as earlier.

Remark 2. *Notice that in the beginning, we may have some of the a_i's be 0; this is analogous to the 0's that occur at the front end of numbers such as 0.003*

Clearly $\left\{ y_m^{(k)} \right\}$ converges to x; $x - y_m^{(k)} = r_m < 10^{-m}$, so for any ϵ you may choose, by picking m large enough, you can make this difference smaller than ϵ. So, we have found one decimal expansion for x.

Is it the only one? Let

(10)
$$x = \sum_{i=k}^{\infty} a_i \cdot 10^{-i}$$

(11)
$$= \sum_{i=k}^{\infty} b_i \cdot 10^{-i}$$

(we can use the same k value for both expressions since we can "pad" one with 0's up front, if need be). Suppose they start differing at position j, and, without loss of generality, $a_j > b_j$. This introduces a difference of $(a_j - b_j) \cdot 10^{-j}$. We now show that even if $a_j - b_k = 1$ (the smallest possible deviation), the only way this difference can be made up in the rest of the series is if $b_i = 9$, $a_i = 0$ for all $i > j$. Indeed, if this extreme case holds holds,

(12)
$$\sum_{i=j+1}^{\infty} (b_i - a_i) \cdot 10^{-i} = 9 \cdot 10^{j+1} \sum_{i'=0}^{\infty} 10^{-i'}$$

(13)
$$= 9 \cdot 10^{j+1} \cdot \frac{10}{9}$$

(14)
$$= 10^j$$

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In other words, we've barely made up the difference in this optimal case; if $a_j - b_j \geq 2$, the rest of the terms have no chance of "catching up." It follows that the only way the decimal expression is not unique is if the first difference that occurs is by 1; then, we must have that in one expansion, say $\sum_{i=k}^{\infty} a_i \cdot 10^{-i}$, $a_i = 0$ for all i past some point. In other words, there exists $n \in \mathbb{N}$ such that $10^n \cdot x \in \mathbb{N}$, which completes the proof.