

Solution Set 8D

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Math 23a

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5 (D) Consider the function $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$ given by $\|(x_1, \dots, x_n)\| = \max\{|x_1|, \dots, |x_n|\}$.

(a) Show that $\|\cdot\|$ defines a norm.

Solution: Let $\mathbf{v} = (x_1, \dots, x_n)$. Then $\|\mathbf{v}\| = |x_i|$ for some $1 \leq i \leq n$. Hence, $|x_i| = \max\{|x_1|, \dots, |x_n|\}$, so $|x_i| \geq |x_j|$ for $1 \leq j \leq n$. Now consider $c \in \mathbb{R}$. Since $|c|$ is non-negative, $|c||x_i| \geq |c||x_j|$ for $1 \leq j \leq n$, and thus, $|cx_i| \geq |cx_j|$ for $1 \leq j \leq n$. Hence, $\|c\mathbf{v}\| = \max\{|cx_1|, \dots, |cx_n|\} = |cx_i|$, so $\|c\mathbf{v}\| = |cx_i| = |c||x_i| = |c|\|\mathbf{v}\|, \forall v \in V, \forall c \in \mathbb{R}$.

Notice as well that as $\|\mathbf{v}\| = |x_i|$ and, by definition, $|x_i| \geq 0, \|\mathbf{v}\| \geq 0, \forall v \in V$. Now assume that $\mathbf{v} = \mathbf{0} = (0, 0, \dots, 0)$. Then all $x_i = 0$, so $\max\{|x_1|, \dots, |x_n|\} = 0$, and hence $\|\mathbf{v}\| = 0$. Moreover, if $\|\mathbf{v}\| = 0$, then $\max\{|x_1|, \dots, |x_n|\} = 0$, so $|x_i| \leq 0$ for $1 \leq i \leq n$. But since $|x_i| \geq 0$ by definition, $|x_i| = 0$, requiring that $x_i = 0$ for $1 \leq i \leq n$. Thus, $\|\mathbf{v}\| = 0$ iff $\mathbf{v} = \mathbf{0}$.

Now consider $\mathbf{u} = (x_1, \dots, x_n)$ and $\mathbf{v} = (y_1, \dots, y_n)$. Then $\|\mathbf{u}\| = |x_i|$ and $\|\mathbf{v}\| = |y_j|$ for some $1 \leq i, j \leq n$, where $|x_i| = \max\{|x_1|, \dots, |x_n|\}$ and $|y_j| = \max\{|y_1|, \dots, |y_n|\}$. Consider $\|\mathbf{u} + \mathbf{v}\| = \max\{|x_1 + y_1|, \dots, |x_n + y_n|\} = |x_k + y_k|$ for some $1 \leq k \leq n$. By the triangle inequality, $|x_k + y_k| \leq |x_k| + |y_k|$, but as $|x_i| = \max\{|x_1|, \dots, |x_n|\}$ and $|y_j| = \max\{|y_1|, \dots, |y_n|\}$, $|x_k| \leq |x_i|$ and $|y_k| \leq |y_j|$. Hence, $\|\mathbf{u} + \mathbf{v}\| = |x_k + y_k| \leq |x_k| + |y_k| \leq |x_i| + |y_j| = \|\mathbf{u}\| + \|\mathbf{v}\|$. Thus, $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|, \forall \mathbf{u}, \mathbf{v} \in V$.

Thus, $\|\cdot\|$ defines a norm.

(b) In Euclidean space, the map $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ is the norm *associated* to the inner product. When $n > 1$, show that there is no inner product $\langle \cdot, \cdot \rangle$ such that this norm is associated to the inner product.

Solution: By contradiction. Assume that this norm is associated to some inner product $\langle \cdot, \cdot \rangle$, and consider $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle + \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle$. Expanding the right hand side of this equation through bilinearity, we have that $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{u} \rangle - 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle = 2\langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{v}, \mathbf{v} \rangle = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$, or, condensing this expression, that $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$. Now let $\mathbf{u} = \mathbf{e}_1$ and $\mathbf{v} = \mathbf{e}_2$. Then $\mathbf{u} + \mathbf{v} = (1, 1, 0, 0, \dots, 0)$ and $\mathbf{u} - \mathbf{v} = (1, -1, 0, 0, \dots, 0)$ so that $\|\mathbf{u} + \mathbf{v}\|^2 = \max\{|1|, |1|, |0|, |0|, \dots, |0|\} = 1$ and $\|\mathbf{u} - \mathbf{v}\|^2 = \max\{|1|, |-1|, |0|, |0|, \dots, |0|\} = 1$. But $\|\mathbf{u}\| = \|\mathbf{v}\| = |1| = 1$, so substituting into equation one yields $1 + 1 = 2(1) + 2(1)$, which is clearly a contradiction, as over \mathbb{R} , $2 \neq 4$.

Hence, this norm can not be associated with any inner product.

Notes: This question proved to be fairly difficult as the contradiction in the second part was not immediately apparent. Just a few comments on common mistakes.

1. The most common errors involved assuming something about the inner product. Some students, for example, treated the inner product as the dot product. While the dot product is certainly a legitimate inner product, the point of the proof is to show that no inner product could possibly be associated with this norm - hence, assuming that a given inner product operated like the dot product reduced generality. Along these lines, many people operated under the assumption that their inner product was of the form $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^t \mathbf{A} \mathbf{v}$ as described in part A of this problem set. While this is a reasonable enough belief (I'm actually not sure if it's true), we never proved that every inner product has this form. Hence, stating this without proof reduced generality.

2. Similarly, some proofs proceeded by creating an orthonormal basis. Certainly, Gram-Schmidt does guarantee us an orthonormal basis, but without further proof, there's no reason to believe that this basis is necessarily the standard one. This is problematic because our norm, by convention, is defined with respect to the standard basis. In actuality, I believe (although I'm not actually sure) that in fact the standard basis is orthogonal with respect to any inner product in \mathbb{R}^n (and in fact some students proved some version of this), but as we never proved this in class, it can not just be assumed. Trying to use Gram-Schmidt thus required operating with two different bases. At any rate, the take home message from the above notes is that while we have not really studied inner products in great depth, and thus do not have a great sense for what facts about them actually are true, you can still solve this proof without assuming or proving additional assertions.

3. Finally, just remember to apply specific examples. Many people, for example, started to generate promising contradictions, but then just stopped by saying that clearly two sides of an equation could not be equal for all vectors. Plug in numbers

and illustrate the contradiction explicitly. Along these lines, prove your assertions explicitly - many people lost points on the first part because they left out important points about $|c|$ being non-negative, about the triangle inequality applying, etc. At this point in the course, it is not really sufficient to make reasonable, but not mathematically rigorous, appeals to the reader's sensibility - claiming something based on "common sense" is not adequate.

4. Despite my above grumbling, proofs really have improved greatly. While questions have become much harder, you all have done a really good job hanging in there and making reasonable arguments. Congratulations, and good luck on the exam.