Math 23a: Theoretical Linear Algebra
and Multivariable Calculus I

FINAL EXAM

January 24, 2006

Your name: ____________________

Problem	Points	Score
1	15	
2	10	
3	10	
4	10	
5	15	
6	20	
7	20	
Total	100	

In the following problems you can use any of the results we have proved in class, if you state them clearly before using them.

Please show all your work on this exam paper. Unless otherwise stated, you must show your work and clearly indicate your line of reasoning in order to get full credit. You can write on the back of the pages if you need extra paper.

Problem 1 [15 points]

Decide whether the following statements are True or False. (Note: There is no need to justify your answers, just circle T or F. You get +3 for every correct answer and -1 for every wrong answer.)

T or F: The set of rational numbers forms an ordered field.

True.

T or F: If A, B are $n \times n$ matrices, $A \neq 0$, and $AB = 0$, then $B = 0$.

False.

T or F: Every orthonormal list of vectors in a finite dimensional Euclidean space V can be extended to an orthonormal basis of V.

True.

T or F: Let $T : V \to V$ be an operator on a finite dimensional Euclidean space V. Eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

False.

T or F: If A is a unitary matrix, then $|\det A| = 1$.

True.

Problem 2 [10 points]

Find a basis for the solution space $U \subset \mathbb{R}^4$ of the following homogeneus system of linear equations:

$$\begin{cases} x_1 - x_2 + 2x_3 - x_4 &= 0 \\ x_2 + x_3 - 3x_4 &= 0 \\ 2x_1 - x_2 + 5x_3 - 5x_4 &= 0 \end{cases}$$

Solution

The matrix associated to the system is

$$\begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & 1 & -3 \\ 2 & -1 & 5 & -5 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & 3 & -4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

hence, the original system is equivalent to

$$\begin{cases} x_1 + 3x_3 - 4x_4 &= 0 \\ x_2 + x_3 - 3x_4 &= 0 \end{cases}$$

We get a basis by picking $x_3 = 1, x_4 = 0$ and $x_3 = 0, x_4 = 1$.

Answer:

Basis U:	$v_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \end{bmatrix}$

Problem 3 [10 points]

Say which of the following functions is a linear transformation and, in case it is, write down the corresponding matrix A, the kernel Kerϕ and the image Imϕ.

(a) $\phi : \mathbb{R}^3 \to \mathbb{R}^2, \qquad \phi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} xy + z \\ yz - x \end{bmatrix}$

(b) $\phi : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ -4y - 2x \end{bmatrix}$

(c) $\phi : \mathbb{R}^3 \to \mathbb{R}^2, \qquad \phi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y + z \\ x - y - 1 \\ z + x \end{bmatrix}$

Answers:

(a)	**N**	A= -	Kerϕ= -	Imϕ= -
(b)	**Y**	A= $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$	Kerϕ= $\left\{\begin{bmatrix} -2\alpha \\ \alpha \end{bmatrix} \middle\vert\, \alpha \in \mathbb{R}\right\}$	Imϕ= $\left\{\begin{bmatrix} \alpha \\ -2\alpha \end{bmatrix} \middle\vert\, \alpha \in \mathbb{R}\right\}$
(c)	**N**	A= -	Kerϕ= -	Imϕ= -

Problem 4 [10 points]

Compute the determinant of the following matrices:

(a) $A = \begin{bmatrix} 0 & 0 & 5 & 1 \\ -1 & -2 & 6 & 2 \\ 0 & 0 & 7 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1001 & 1002 & 1003 & 1004 \\ 1002 & 1003 & 1001 & 1002 \\ 1001 & 1001 & 1001 & 999 \\ 1001 & 1001 & 998 & 996 \end{bmatrix}$

Solution

$\det A$ is zero since column 2 is twice column 1. For $\det B$ one needs to perform a computation:

$$\begin{aligned} \det B &= \begin{vmatrix} 1001 & 1002 & 1003 & 1004 \\ 1002 & 1003 & 1001 & 1002 \\ 1001 & 1001 & 1001 & 999 \\ 1001 & 1001 & 998 & 996 \end{vmatrix} = \begin{vmatrix} 1001 & 1002 & 1003 & 1004 \\ 1 & 1 & -2 & -2 \\ 0 & -1 & -2 & -5 \\ 0 & -1 & -5 & -8 \end{vmatrix} \\ &= 1001 \begin{vmatrix} 1 & -2 & -2 \\ -1 & -2 & -5 \\ -1 & -5 & -8 \end{vmatrix} - \begin{vmatrix} 1002 & 1003 & 1004 \\ -1 & -2 & -5 \\ -1 & -5 & -8 \end{vmatrix} \\ &= 1001(-9) - (-9 \cdot 1002 - 3 \cdot 1003 + 3 \cdot 1004) \\ &= -9009 + 9018 + 3009 - 3012 \; = \; 6 \end{aligned}$$

Answers:

$\det A =$	0
$\det B =$	6

Problem 5 [15 points]

Suppose V is a vector space over $\mathbb{C}$ of dimension n and $T : V \to V$ is a linear operator on V. Suppose that T has only one eigenvalue λ. Prove that $(T - \lambda \mathbb{1})^n = 0$.

Solution

We know there is a basis in which the matrix A of T is upper triangular with the eigenvalues on the diagonal. By assumption λ is the only eigenvalue, so A is upper triangular with λ on the diagonal. Hence the matrix for $T - \lambda \mathbb{1}$ is $A - \lambda \mathbb{1}$, which is upper triangular with zero on the diagonal. But it was proved in a problem set that such a matrix satisfies $(A - \lambda \mathbb{1})^n = 0$.

Problem 6 [20 points]

Let A be an $n \times n$ Hermitian matrix with complex coefficients, and denote by a_{ij} the entry of A in row i and column j. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, counted with multiplicities.

(a) State the Spectral Theorem for Hermitian matrices.

(b) Let λ be the lowest eigenvalue: $\lambda = \min\{\lambda_1, \ldots, \lambda_n\}$. Prove that $a_{ii} \geq \lambda$ for every $i = 1, \ldots, n$.

Solution

(a) Any Hermitian matrix admits an orthonormal basis of eigenvectors.

(b) Call $(e_1, \ldots, e_n)$ the standard basis in $\mathbb{C}^n$, and $(v_1, \ldots, v_n)$ be an orthonormal basis of eigenvectors for A (it exists by the Spectral Theorem). We can write each basis vector e_i as linear combination of the v_i's:

$$e_i \;=\; \langle e_i|v_1\rangle v_1 + \langle e_i|v_2\rangle v_2 + \cdots + \langle e_i|v_n\rangle v_n \;.$$

Recall that the matrix elements of A are given by $a_{ij} = \langle Ae_j|e_i\rangle$. Hence we have

$$\begin{aligned}
a_{ii} &= \langle Ae_i|e_i\rangle \\
&= \Big\langle A\Big(\langle e_i|v_1\rangle v_1 + \cdots + \langle e_i|v_n\rangle v_n\Big)\Big|\langle e_i|v_1\rangle v_1 + \cdots + \langle e_i|v_n\rangle v_n\Big\rangle \\
&= \Big\langle \lambda_1\langle e_i|v_1\rangle v_1 + \cdots + \lambda_n\langle e_i|v_n\rangle v_n\Big|\langle e_i|v_1\rangle v_1 + \cdots + \langle e_i|v_n\rangle v_n\Big\rangle \\
&= \lambda_1\langle e_i|v_1\rangle\overline{\langle e_i|v_1\rangle} + \lambda_2\langle e_i|v_2\rangle\overline{\langle e_i|v_2\rangle} + \cdots + \lambda_n\langle e_i|v_n\rangle\overline{\langle e_i|v_n\rangle} \\
&\geq \lambda\Big(\langle e_i|v_1\rangle\langle v_1|e_i\rangle + \langle e_i|v_2\rangle\langle v_2|e_i\rangle + \cdots + \langle e_i|v_n\rangle\langle v_n|e_i\rangle\Big) \\
&= \lambda\langle e_i|e_i\rangle \;=\; \lambda
\end{aligned}$$

Problem 7 [20 points]

Consider the field of two elements $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Let S be any set, and let V_S be the set consisting of all subsets of S. We define on V_S two operations: addition

$$A + B \ = \ (A - B) \cup (B - A)$$

(recall that $A - B$, the difference of subsets, denotes the set of all elements of A which are not in B), and scalar multiplication (by elements of $\mathbb{Z}/2\mathbb{Z}$):

$$0 \cdot A \ = \ \emptyset \ , \quad 1 \cdot A \ = \ A \ .$$

(a) I claim that V_S, with the above operations, is a vector space over $\mathbb{Z}/2\mathbb{Z}$. You do not need to check all the axioms. Just state and prove the following vector space axioms:
   1. associativity of addition (what is $A + B + C$?),
   2. existence of additive identity,
   3. existence of additive inverse

(b) Given a subset $A \subset S$, consider the following functions I_A, $U_A : V_S \to V_S$,

$$I_A(B) = A \cap B \ , \quad U_A(B) = A \cup B \ .$$

Say whether the above functions are linear transformations on V_S and, if they are, determine their kernel and their image.

Solution

(a) Of course one can check directly that $(A+B)+C = A+(B+C) =$ the set of all elements in either only one or all three of A, B, C (it's immediate to see with a picture), that $\emptyset + A = A$ and that $A + A = \emptyset$.

Another way to solve this problem is to notice that V_S is isomorphic to the vector space $(\mathbb{Z}/2\mathbb{Z})^S$, namely a vector space over $\mathbb{Z}/2\mathbb{Z}$ with basis S. (a subset $A \subset S$ to be identified with the "column vector" with entry 1 in correspondence of elements of A and 0 in correspondence of all elements in $S - A$). Then the sum and scalar multiplication defined above are just the addition and scalar multiplication of column vectors in $(\mathbb{Z}/2\mathbb{Z})^S$ (it's immediate to check).

(b) I_A is a linear map, since (each equality is immediate to check) $I_A(B+C) = A \cap ((B - C) \cup (C - B)) = (A \cap (B - C)) \cup (A \cap (C - B)) = ((A \cap B) - C) \cup ((A \cap C) - B) = ((A \cap B) - (A \cap C)) \cup ((A \cap C) - (A \cap B)) = (A \cap B) + (A \cap C) = I_A(B) + I_A(C)$, $I_A(0 \cdot B) = I_A(\emptyset) = A \cap \emptyset = \emptyset = 0 \cdot I_A(B)$ and $I_A(1 \cdot B) = I_A(B) = 1 \cdot I_A(B)$. U_A is not linear since $U_A(0 \cdot B) = U_A(\emptyset) = A \cup \emptyset = A \neq \emptyset = 0 \cdot U_A(B)$. Finally, $\ker I_A = \{B \,|\, A \cap B = \emptyset\} = \{B \subset S - A\}$ and $\mathrm{Im} I_A = \{A \cap B \,|\, B \subset S\} = \{B \subset A\}$.