Math 23a: Theoretical Linear Algebra
and Multivariable Calculus I

MIDTERM EXAM 1

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Problem	Points	Score
1	20	20
2	20	20
3	20	20
4	20	20
5	20	20
Total	100	100

In the following problems you can use any of the results we have proved in class, if you state them clearly before using them.

Please show all your work on this exam paper. You must show your work and clearly indicate your line of reasoning in order to get full credit. If you have work on the back of a page, indicate that on the exam cover.

Problem 1

Let V be a vector space and let $U \subset V$ be a subspace. Consider the following relation $\sim_U$ of V:

$$v_1 \ \sim_U \ v_2 \quad \text{if and only if} \quad v_1 - v_2 \in U \ .$$

(a) Prove that $\sim_U$ is an equivalence relation.
(b) Describe the corresponding partition (what are its elements?)

Solution

(a) To show that $\sim$ is an equivalence relation we need to show it is reflexive, symmetric and transitive.
reflexive: Since U is a subspace, we know $0 \in U$, hence $v - v = 0 \in U$, $\forall v \in V$, which means $v \sim v$, $\forall v \in V$
symmetric: If $v_1 \sim v_2$, then $v_1 - v_2 \in U$. Since U is a subspace (in particular, closed under scalar multiplication) we then have $v_2 - v_1 = (-1)(v_1 - v_2) \in U$, which implies $v_2 \sim v_1$.
transitive: If $v_1 \sim v_2$ and $v_2 \sim v_3$, then $v_1 - v_2 \in U$ and $v_2 - v_3 \in U$. Since U is a subspace (in particular, closed under addition) we then have $v_1 - v_3 = (v_1 - v_2) + (v_2 - v_3) \in U$, which implies $v_1 \sim v_3$.

(b) The corresponding partition is

$$V/\sim = \Big\{ [v] = \big\{ v + u \mid u \in U \big\} \ \Big| \ v \in V \Big\} \ .$$

Problem 2

Consider the following set: $\mathbb{Q}[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$. (Namely, the elements of this set are symbols "$a + b\sqrt{3}$" where a and b are rational numbers). We define the following operations of addition and multiplication on $\mathbb{Q}[\sqrt{3}]$:

$$\begin{aligned}(a + b\sqrt{3}) + (c + d\sqrt{3}) &= (a + c) + (b + d)\sqrt{3} \ , \\ (a + b\sqrt{3}) \cdot (c + d\sqrt{3}) &= (ac + 3bd) + (ad + bc)\sqrt{3} \ .\end{aligned}$$

It is a fact that $\mathbb{Q}[\sqrt{3}]$ with these operations is a field. I don't ask you to prove all the axioms of field, only a part of them.

(a) What do you think should be the elements 0 and 1 in this field?
(b) State axiom 5 (existence of negatives) and axiom 6 (existence of reciprocals) of fields.
(c) Prove that axiom 5 and axiom 6 hold in $\mathbb{Q}[\sqrt{3}]$

(You can use the fact that $\sqrt{3} \notin \mathbb{Q}$)

Solution

(a) $0_{\mathbb{F}} = 0 + 0\sqrt{3}$, $1_{\mathbb{F}} = 1 + 0\sqrt{3}$.

(b) *Axiom 5:* For every $A \in \mathbb{F}$, there exist some element $(-A) \in \mathbb{F}$ such that $A + (-A) = 0_{\mathbb{F}}$.
Axiom 6: For every non zero $A \in \mathbb{F}$, there exist some element $A^{-1} \in \mathbb{F}$ such that $A \cdot A^{-1} = 1_{\mathbb{F}}$.

(c) If $A = a + b\sqrt{3}$, we can take $(-A) = (-a) + (-b)\sqrt{3}$, indeed, by definition of addition,

$$\begin{aligned}A + (-A) &= (a + b\sqrt{3}) + ((-a) + (-b)\sqrt{3}) \\ &= (a + (-a)) + (b + (-b))\sqrt{3} = 0 + 0\sqrt{3} = 0_{\mathbb{F}} \ .\end{aligned}$$

If $A = a + b\sqrt{3} =\neq 0_{\mathbb{F}}$ (i.e. either $a \neq 0$ or $b \neq 0$), we can take $A^{-1} = \frac{a}{a^2-3b^2} + \frac{-b}{a^2-3b^2}\sqrt{3}$. First of all, notice that this expression of A^{-1} makes sense. Indeed, since $a, b \in \mathbb{Q}$, then $a/b \in \mathbb{Q}$, so $a/b \neq \sqrt{3}$ (because $\sqrt{3} \notin \mathbb{Q}$), hence $a^2 - 3b^2 \neq 0$. Morevoer, by definition of multiplication we have

$$\begin{aligned}&A \cdot A^{-1} = (a + b\sqrt{3}) \cdot (\frac{a}{a^2 - 3b^2} + \frac{-b}{a^2 - 3b^2}\sqrt{3}) \\ &= (a\frac{a}{a^2 - 3b^2} + 3b\frac{-b}{a^2 - 3b^2}) + (a\frac{-b}{a^2 - 3b^2} + b\frac{a}{a^2 - 3b^2})\sqrt{3} = 1 + 0\sqrt{3} = 1_{\mathbb{F}} \ .\end{aligned}$$

Problem 3

Let U_1 and U_2 be subspaces of a vector space V. Prove or disprove:

(a) $U_1 \cap U_2$ is a subspace of V,
(b) $U_1 \cup U_2$ is a subspace of V.

(Note: to "disprove" something means to find a counter-example)

Solution

(a) $U_1 \cap U_2$ is a subspace. Indeed, first of all $0 \in U_1 \cap U_2$, hence $U_1 \cap U_2$ is not empty. If $\lambda \in \mathbb{F}$ and $u \in U_1 \cap U_2$, which means $u \in U_1$ and $u \in U_2$, then $\lambda u \in U_1$ (because U_1 is a subspace) and $\lambda u \in U_2$ (because U_2 is a subspace), hence $\lambda u \in U_1 \cap U_2$. Moreover, if u and v belong to $U_1 \cap U_2$, namely u and v belong to both U_1 and U_2, then we have $u + v \in U_1$ (because U_1 is a subspace) and $u + v \in U_2$ (because U_2 is a subspace), hence $u + v \in U_1 \cap U_2$. This shows that $U_1 \cap U_2$ is a subspace.

(b) $U_1 \cup U_2$ is not a subspace. Here is counter-example.

$$U_1 = \left\{ \left[\begin{array}{c} \alpha \\ 0 \end{array} \right] \middle| \alpha \in \mathbb{R} \right\} \ , \ \ U_2 = \left\{ \left[\begin{array}{c} 0 \\ \alpha \end{array} \right] \middle| \alpha \in \mathbb{R} \right\} \ \subset \ \mathbb{R}^2 \ .$$

Indeed

$$a = \left[\begin{array}{c} 1 \\ 0 \end{array} \right] \in U_1 \subset U_1 \cup U_2 \ , \ \ b = \left[\begin{array}{c} 0 \\ 1 \end{array} \right] \in U_2 \subset U_1 \cup U_2 \ ,$$

but

$$a + b = \left[\begin{array}{c} 1 \\ 1 \end{array} \right] \notin U_1 \cup U_2 \ ,$$

hence $U_1 \cup U_2$ is not closed under addition, and therefore it is not a subspace.

Problem 4

Let $V = (\mathbb{Z}/7\mathbb{Z})^3$, and consider the following vectors of V:

$$a = \begin{bmatrix} [1] \\ [0] \\ [6] \end{bmatrix} \;,\quad b = \begin{bmatrix} [1] \\ [2] \\ [1] \end{bmatrix} \;,\quad c = \begin{bmatrix} [2] \\ [1] \\ [6] \end{bmatrix} \;.$$

Prove that they are linearly dependent.

Solution

Here is a relation of linear dependence:

$$\begin{aligned} [5]a + [4]b + [6]c &= [5] \begin{bmatrix} [1] \\ [0] \\ [6] \end{bmatrix} + [4] \begin{bmatrix} [1] \\ [2] \\ [1] \end{bmatrix} + [6] \begin{bmatrix} [2] \\ [1] \\ [6] \end{bmatrix} \\ &= \begin{bmatrix} [21] \\ [14] \\ [70] \end{bmatrix} = \begin{bmatrix} [0] \\ [0] \\ [0] \end{bmatrix} = 0 \end{aligned}$$

Problem 5

Prove the following statements:

(a) Suppose $(v_1, \dots, v_n)$ is a basis of the vector space V. Then $(v_1 - v_2, v_2 - v_3 \dots, v_{n-1} - v_n, v_n)$ is a basis of V.

(b) Suppose $(v_1, \dots, v_n)$ are linearly independent vectors of V, and suppose $w \in V$ is such that $(v_1 + w, v_2 + w, \dots, v_n + w)$ are linearly dependent. Then $w \in \operatorname{span}(v_1, \dots, v_n)$.

Solution

(a) First, let us prove that $(v_1 - v_2, v_2 - v_3 \dots, v_{n-1} - v_n, v_n)$ is linearly independent. Suppose then

$$\lambda_1(v_1 - v_2) + \lambda_2(v_2 - v_3) + \cdots + \lambda_{n-1}(v_{n-1} - v_n) + \lambda_n v_n = 0 \ ,$$

and we need to prove that all the λ_i's are zero. The above equation is equivalent to

$$\lambda_1 v_1 + (\lambda_2 - \lambda_1)v_2 + \cdots + (\lambda_{n-1} - \lambda_{n-2})v_{n-1} + (\lambda_n - \lambda_{n-1})v_n = 0 \ ,$$

By assumption, $(v_1, \dots, v_n)$ are linearly independent, hence the above equation implies

$$\lambda_1 = 0, \ \lambda_2 - \lambda_1 = 0, \dots, \lambda_{n-1} - \lambda_{n-2} = 0, \ \lambda_n - \lambda_{n-1} = 0 \ ,$$

which is equivalent to say all λ_i's are zero.

We then have to show $(v_1 - v_2, v_2 - v_3 \dots, v_{n-1} - v_n, v_n)$ span the whole vector space V. Let then $v \in V$, and we want to write it as a linear combination of $(v_1 - v_2), (v_2 - v_3), \dots, (v_{n-1} - v_n), v_n$. By assumption, $(v_1, \dots, v_n)$ is a basis, so we can find $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ such that

$$v = \lambda_1 v_1 + \cdots + \lambda_{n-1} v_{n-1} + \lambda_n v_n \ .$$

The above equation implies

$$v = \lambda_1(v_1 - v_2) + (\lambda_1 + \lambda_2)(v_2 - v_3) + \cdots + (\lambda_1 + \cdots + \lambda_{n-1})(v_{n-1} - v_n) \\ +(\lambda_1 + \cdots + \lambda_{n-1} + \lambda_n)v_n \ ,$$

which is the relation we wanted.

(b) By assumption, we can find $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ not all zero such that

$$\lambda_1(v_1 + w) + \cdots + \lambda_n(v_n + w) = 0 \ .$$

The above equation is equivalent to

$$\lambda_1 v_1 + \cdots + \lambda_n v_n + (\lambda_1 + \cdots + \lambda_n)w = 0 \ .$$

Notice that it can't be $\lambda_1 + \cdots + \lambda_n = 0$, otherwise the above equation would say that $(v_1, \dots, v_n)$ are linearly dependent, which is not true by assumption. So we have $\Lambda = \lambda_1 + \cdots + \lambda_n \neq 0$. Hence by the above equation we get

$$w = \frac{-\lambda_1}{\Lambda} v_1 + \cdots + \frac{-\lambda_n}{\Lambda} v_n \ ,$$

which proves the statement.