Math 23a: Theoretical Linear Algebra
and Multivariable Calculus I

MIDTERM EXAM 1

October 17, 2005

Your name: ____________________

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

In the following problems you can use any of the results we have proved in class, if you state them clearly before using them.

Please show all your work on this exam paper. You must show your work and clearly indicate your line of reasoning in order to get full credit. If you have work on the back of a page, indicate that on the exam cover.

Problem 1

Let V be a vector space and let $U \subset V$ be a subspace. Consider the following relation $\sim_U$ of V:

$$v_1 \sim_U v_2 \quad \text{if and only if} \quad v_1 - v_2 \in U \ .$$

(a) Prove that $\sim_U$ is an equivalence relation.
(b) Describe the corresponding partition (what are its elements?)

Problem 2

Consider the following set: $\mathbb{Q}[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$. (Namely, the elements of this set are symbols "$a + b\sqrt{3}$" where a and b are rational numbers). We define the following operations of addition and multiplication on $\mathbb{Q}[\sqrt{3}]$:

$$(a + b\sqrt{3}) + (c + d\sqrt{3}) = (a + c) + (b + d)\sqrt{3} ,$$
$$(a + b\sqrt{3}) \cdot (c + d\sqrt{3}) = (ac + 3bd) + (ad + bc)\sqrt{3} .$$

It is a fact that $\mathbb{Q}[\sqrt{3}]$ with these operations is a field. I don't ask you to prove all the axioms of field, only a part of them.

(a) What do you think should be the elements 0 and 1 in this field?

(b) State axiom 5 (existence of negatives) and axiom 6 (existence of reciprocals) of fields.

(c) Prove that axiom 5 and axiom 6 hold in $\mathbb{Q}[\sqrt{3}]$

(You can use the fact that $\sqrt{3} \notin \mathbb{Q}$)

Problem 3

Let U_1 and U_2 be subspaces of a vector space V. Prove or disprove:

(a) $U_1 \cap U_2$ is a subspace of V,

(b) $U_1 \cup U_2$ is a subspace of V.

(Note: to "disprove" something means to find a counter-example)

Problem 4

Let $V = (\mathbb{Z}/7\mathbb{Z})^3$, and consider the following vectors of V:

$$a = \begin{bmatrix} [1] \\ [0] \\ [6] \end{bmatrix}, \quad b = \begin{bmatrix} [1] \\ [2] \\ [1] \end{bmatrix}, \quad c = \begin{bmatrix} [2] \\ [1] \\ [6] \end{bmatrix}.$$

Prove that they are linearly dependent.

Problem 5

Prove the following statements:

(a) Suppose $(v_1, \ldots, v_n)$ is a basis of the vector space V. Then $(v_1 - v_2, v_2 - v_3 \ldots, v_{n-1} - v_n, v_n)$ is a basis of V.

(b) Suppose $(v_1, \ldots, v_n)$ are linearly independent vectors of V, and suppose $w \in V$ is such that $(v_1 + w, v_2 + w, \ldots, v_n + w)$ are linearly dependent. Then $w \in \operatorname{span}(v_1, \ldots, v_n)$.