Math 23a: Theoretical Linear Algebra
and Multivariable Calculus I

MIDTERM EXAM 2 - Solutions

October 17, 2005

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Problem	Points	Score
1	20	20
2	20	20
3	20	20
4	20	20
5	20	20
Total	100	100

In the following problems you can use any of the results we have proved in class, if you state them clearly before using them.

Please show all your work on this exam paper. Unless otherwise stated, you must show your work and clearly indicate your line of reasoning in order to get full credit. You can write on the back of the pages if you need extra paper.

Problem 1

Decide whether the following statements are True or False. (Note: There is no need to justify your answers. You get +4 for every correct answer and -1 for every wrong answer.)

T or F: If V is a finite-dimensional vector space spanned by a set of n vectors, then $\dim V = n$.

False: they can be linearly dependent.

T or F: If $(v_1, \ldots, v_n)$ is a set of vectors such that none of them is a scalar multiple of any others, then they are linearly independent.

False: the vectors $[1, 0]$, $[0, 1]$, $[1, 1]$ are not linearly independent, and yet none of them is scalar multiple of any other.

T or F: $\lambda = 0$ is an eigenvalue of a linear transformation $T : V \to V$ (on a finite dimensional vector space V) if and only if T is not injective.

True: T is not injective, iff $\mathrm{Ker}(T) \neq 0$, iff there exists $v \neq 0$ such that $T(v) = 0$, iff there is an eigenvector of eigenvalue 0, iff 0 is eigenvalue.

T or F: If A is an invertible 3×3 matrix with real coefficients, then its rows form a basis of $\mathbb{R}^3$.

True: This, I think, was in one of the problem sets!

T or F: $sign(\sigma) = 1$ where $\sigma \in S_5$ is (with the notation introduced in class)

$$\begin{pmatrix} 1,2,3,4,5 \\ 3,5,1,4,2 \end{pmatrix}$$

True: to go from σ to the identity permutation it suffices to switch 1&3, 2&5, so $sign(\sigma) = 1$.

Problem 2

Let L be the vector space of all infinite sequences of real numbers:

$$L = \left\{ (a_1, a_2, a_3, \dots) \;\middle|\; a_i \in \mathbb{R},\, i \in \mathbb{N} \right\} .$$

(with scalar multiplication and addition defined entry by entry). Let S be the "shift operator" of L, defined by

$$S(a_1, a_2, a_3, \dots) \;=\; (a_2, a_3, a_4, \dots) \,.$$

and consider the linear transformation $T : L \to L$ defined by $T = S \circ S \circ S - S \circ S - S$.

(a) For a generic vector $(a_1, a_2, a_3, a_4, \dots)$ write down explicitly $T(a_1, a_2, a_3, a_4, \dots)$.

(b) Find a basis of $\mathrm{Ker}(T)$ and compute $\dim \mathrm{Ker}(T)$.

Answer:

(a) $T(a_1, a_2, a_3, a_4, \dots) = (S \circ S \circ S - S \circ S - S)(a_1, a_2, a_3, a_4, \dots) = S(S(S(a_1, a_2, a_3, a_4, \dots))) - S(S(a_1, a_2, a_3, a_4, \dots)) - S(a_1, a_2, a_3, a_4, \dots) = (a_4 - a_3 - a_2, a_5 - a_4 - a_3, a_6 - a_5 - a_4, \dots)$

(b) $\mathrm{Ker}T = \{(a_1, a_2, a_3, a_4, \dots) \,|\, a_{n+2} = a_{n+1} + a_n,\ \forall n \geq 2\}$. A basis for this space is $\mathcal{B} = \Big((1,0,0,0,0,\dots),\ (0,1,0,1,1,2,3,5,\dots),\ (0,0,1,1,2,3,5,8,\dots) \Big)$. These vectors are clearly linearly independent, and it's immediate to check that they span $\mathrm{Ker}T$. Hence $\dim \mathrm{Ker}T = 3$.

Answers:

$T(a_1, a_2, a_3, a_4, \dots) =$	$(a_4 - a_3 - a_2, a_5 - a_4 - a_3, a_6 - a_5 - a_4, \dots)$
Basis of $\mathrm{Ker}(T)$:	$(1,0,0,0,0,\dots)$, $(0,1,0,1,1,2,3,5,\dots)$, $(0,0,1,1,2,3,5,8,\dots)$
$\dim \mathrm{Ker}(T) =$	3

Problem 3

Consider the following three dimensional subspace of the space of all real functions:

$$V = \left\{ f(x) = ae^x + b\sqrt{x} + c\sin x \;\middle|\; a,b,c \in \mathbb{R} \right\} .$$

Let $T : V \to V$ be the operator of V which has the following matrix in basis $(v_1 = e^x,\ v_2 = \sqrt{x},\ v_3 = e^x - \sqrt{x} + \sin x)$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} .$$

(a) Compute $T(\sin x)$.
(b) Write the formula for the inverse matrix A^{-1} of an arbitrary matrix A.
(c) Use this formula to prove that the matrix A^{-1} is upper triangular with 1 on the diagonal, namely it's of the form

$$A^{-1} = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} .$$

(You don't need to compute A^{-1} completely)
(d) Compute $T^{-1}(e^x)$.

Answer:

(a) $\sin(x) = -e^x + \sqrt{x} + (e^x - \sqrt{x} + \sin x) = -v_1 + v_2 + v_3$, hence $T(\sin x)$ corresponds (under the isomorphism $V \simeq \mathbb{R}^3$), to

$$A \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

which in turn corresponds to $T(\sin x) = v_1 + 2v_2 + v_3 = e^x + 2\sqrt{x} + (e^x - \sqrt{x} + \sin x) = 2e^x + \sqrt{x} + \sin x$.

(b) $A^{-1} = \frac{1}{\det A} B$ where B is the matrix which, in position (i,j), has entry

$$b_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1,i+1} & \cdots & a_{1n} \\ & \cdots & & & \cdots & \\ a_{j-1,1} & \cdots & a_{j-1,i-1} & a_{j-1,i+1} & \cdots & a_{j-1,n} \\ a_{j+1,1} & \cdots & a_{j+1,i-1} & a_{j+1,i+1} & \cdots & a_{j+1,n} \\ & \cdots & & & \cdots & \\ a_{n1} & \cdots & a_{n,i-1} & a_{n,i+1} & \cdots & a_{nn} \end{vmatrix}$$

(c) In this case, $\det A = 1$, so the entry on the diagonal and below in A^{-1} it are:

$$(11) : \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1 \ , \quad (21) : -\det \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 0$$

$$(22) : \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1 \ , \quad (31) : -\det \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

$$(32) : -\det \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0 \ , \quad (33) : \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1$$

Hence A^{-1} is upper triangular with 1s on the diagonal.

(d) $e^x = v_1$, hence $T^{-1}(e^x)$ corresponds (under the isomorphism $V \simeq \mathbb{R}^3$), to

$$A^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

which in turn corresponds to $T(e^x) = e^x$.

Answers:

$T(\sin x) =$	$2e^x + \sqrt{x} + \sin x$
$T^{-1}(e^x) =$	e^x

Problem 4

Let V be a finite dimensional vector space over $\mathbb{R}$. Let $S : V \to V$ and $T : V \to V$ be linear transformations. Suppose that $ST = TS$ and suppose that $v \notin \mathrm{Ker}T$ is an eigenvector of S with eigenvalue λ.

(a) Prove that Tv is an eigenvector of S (with which eigenvalue?).

(b) Prove that, if $\dim \mathrm{Ker}(S - \lambda \mathbb{1}) = 1$, then v is also an eigenvector of T.

Answer:

(a) $S(T(v)) = T(S(v)) = T(\lambda v) = \lambda T(v)$. Hence $T(v)$ is eigenvector of S with eigenvalue λ.

(b) By assumption, $\mathrm{Ker}(S - \lambda \mathbb{1}) = \mathrm{span}(v)$. Moreover, by (a), $T(v) \in \mathrm{Ker}(S - \lambda \mathbb{1})$. Hence $T(v) = \alpha v$ for some $\alpha \in \mathbb{F}$, namely v is eigenvector of T.

Problem 5

Let V be a finite dimensional vector space over $\mathbb{F}$ with $\dim V = n$. Let $(v_1, \ldots, v_n)$ be a basis of V. We define $V^* = L(V, \mathbb{F})$ to be the vector space of all linear transformation from the vector space V to the vector space $\mathbb{F}$.

(a) What is the dimension of V^*?

(b) Find a basis of V^*.

(c) Prove that V and V^* are isomorphic vector spaces.

Answer:

(a) The space of linear transformations from V to $\mathbb{F}$ is isomorphic to the space of linear transformations from $\mathbb{F}^n$ to $\mathbb{F}$, which is isomorphic to the space of $1 \times n$ matrices, i.e. to the space of n-row vectors, and we know this has dimension n.

(b) A basis of V^* is $(f_1, f_2, \ldots, f_n)$, where $f_i(\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n) = \lambda_i$. Indeed, all functions f_i are clearly linear transformations. Moreover, they are linearly independent, because if $\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n = 0$, then for every $j = 1, \ldots, n$ we have $0 = (\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n)(v_j) = \alpha_j$. Finally, they generate V^*, since if $f : V \to \mathbb{F}$ is a linear transformation, then $f = f(v_1)f_1 + f(v_2)f_2 + \cdots + f(v_n)f_n$.

(c) V and V^* are isomorphic since they have the same dimension, so $V \simeq \mathbb{F}^n \simeq V^*$.

Answers:

$\dim V^* =$	n
Basis of V^*:	$(f_1, f_2, \ldots, f_n)$, where $f_i(\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n) = \lambda_i$