Math 23a: Theoretical Linear Algebra
and Multivariable Calculus I

MIDTERM EXAM 2

October 17, 2005

Your name: ____________________

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

In the following problems you can use any of the results we have proved in class, if you state them clearly before using them.

Please show all your work on this exam paper. Unless otherwise stated, you must show your work and clearly indicate your line of reasoning in order to get full credit. You can write on the back of the pages if you need extra paper.

Problem 1

Decide whether the following statements are True or False. (Note: There is no need to justify your answers. You get +4 for every correct answer and -1 for every wrong answer.)

T or F: If V is a finite-dimensional vector space spanned by a set of n vectors, then $\dim V = n$.

T or F: If $(v_1, \ldots, v_n)$ is a set of vectors such that none of them is a scalar multiple of any others, then they are linearly independent.

T or F: $\lambda = 0$ is an eigenvalue of a linear transformation $T : V \to V$ (on a finite dimensional vector space V) if and only if T is not injective.

T or F: If A is an invertible 3×3 matrix with real coefficients, then its rows form a basis of $\mathbb{R}^3$.

T or F: $sign(\sigma) = 1$ where $\sigma \in S_5$ is (with the notation introduced in class)

$$\begin{pmatrix} 1, 2, 3, 4, 5 \\ 3, 5, 1, 4, 2 \end{pmatrix}$$

Problem 2

Let L be the vector space of all infinite sequences of real numbers:

$$L = \left\{(a_1, a_2, a_3, \dots) \,\middle|\, a_i \in \mathbb{R},\, i \in \mathbb{N}\right\} .$$

(with scalar multiplication and addition defined entry by entry). Let S be the "shift operator" of L, defined by

$$S(a_1, a_2, a_3, \dots) \;=\; (a_2, a_3, a_4, \dots) .$$

and consider the linear transformation $T : L \to L$ defined by $T = S \circ S \circ S - S \circ S - S$.

(a) For a generic vector $(a_1, a_2, a_3, a_4, \dots)$ write down explicitly $T(a_1, a_2, a_3, a_4, \dots)$.

(b) Find a basis of $\mathrm{Ker}(T)$ and compute $\dim \mathrm{Ker}(T)$.

Answers:

$T(a_1, a_2, a_3, a_4, \dots) =$	
Basis of $\mathrm{Ker}(T)$:	
$\dim \mathrm{Ker}(T) =$	

Problem 3

Consider the following three dimensional subspace of the space of all real functions:

$$V = \left\{ f(x) = ae^x + b\sqrt{x} + c\sin x \;\middle|\; a, b, c \in \mathbb{R} \right\} .$$

Let $T : V \to V$ be the operator of V which has the following matrix in basis $(v_1 = e^x,\ v_2 = \sqrt{x},\ v_3 = e^x - \sqrt{x} + \sin x)$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} .$$

(a) Compute $T(\sin x)$.

(b) Write the formula for the inverse matrix A^{-1} of an arbitrary matrix A.

(c) Use this formula to prove that the matrix A^{-1} is upper triangular with 1 on the diagonal, namely it's of the form

$$A^{-1} = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} .$$

(You don't need to compute A^{-1} completely)

(d) Compute $T^{-1}(e^x)$.

Answers:

$T(\sin x) =$	
$T^{-1}(e^x) =$	

Problem 4

Let V be a finite dimensional vector space over $\mathbb{R}$. Let $S : V \to V$ and $T : V \to V$ be linear transformations. Suppose that $ST = TS$ and suppose that v is an eigenvector of S with eigenvalue λ.

(a) Prove that Tv is an eigenvector of S (with which eigenvalue?).

(b) Prove that, if $\dim \operatorname{Ker}(S - \lambda \mathbb{1}) = 1$, then v is also an eigenvector of T.

Problem 5

Let V be a finite dimensional vector space over $\mathbb{F}$ with $\dim V = n$. Let $(v_1, \ldots, v_n)$ be a basis of V. We define $V^* = L(V, \mathbb{F})$ to be the vector space of all linear transformation from the vector space V to the vector space $\mathbb{F}$.

(a) What is the dimension of V^*?

(b) Find a basis of V^*.

(c) Prove that V and V^* are isomorphic vector spaces.

Answers:

$\dim V^* =$	
Basis of V^*:	