

MATHEMATICAL INDUCTION

MATH 23A SECTION HANDOUT 2
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Imagine a bunch of dominoes on a table. They are set up in a straight line, and you are about to push the first piece to set off the chain reaction that will bring all the dominoes down. For this chain reaction to knock off every piece, you have to make sure that every piece is close enough to the next one. Otherwise, the reaction will stop. First, you have to be absolutely sure to have put all the domino pieces in such a way that each of them, while falling, will knock the next one down. Next, you just need to knock down the first piece. You can then be sure that, eventually, every piece is definitely going to fall. Let's turn this into math.

The domino scenario we described above is a surprisingly accurate representation of a proof technique called *mathematical induction*. In this case, the *proof by induction* consists in making sure that the first domino piece falls and that if a domino piece falls, then it will knock the next one off. Then, by *mathematical induction*, we will be sure that every single domino piece will fall. More formally, this technique is summarized below.

Mathematical Induction: Let $A(n)$ be an assertion involving an integer n . If we can perform the following two steps:

- (1) prove that $A(1)$ is true,
- (2) for a given arbitrary k , assume that $A(k)$ is true, and prove that $A(k + 1)$ is also true,

then we can conclude that the assertion $A(n)$ is true for every positive integer n .

In our domino analogy, $A(n)$ is the statement that the n th piece of domino falls. In addition, the first domino getting knocked off refers to the statement $A(1)$ is true and the condition that if a domino falls, then the next one falls as well is analogous to "if $A(x)$ is true, then $A(x + 1)$ is true as well." Finally, the idea of all the dominoes falling is equivalent to the idea of $A(n)$ being true for all positive integers n .

Now we will show an example of a proof by induction. Please note how the different steps are clearly marked.

Proposition 1. *For every positive integer n , the following identity holds.*

$$1 + 2 + \dots + n = \frac{n(n + 1)}{2}.$$

Proof. A proof by induction consists of three different steps. We will label these steps along the proof for instructive purposes and will show an example without the annotations in the next pages.

Base Step (Show that the first domino piece falls) Note that if $n = 1$, then

$1 = \frac{1(1+1)}{2}$ equals the sum of the numbers from 1 to $n = 1$.

In this step, we make sure that whatever we want to prove works in the simplest case of all. Sometimes the base step is one of the hardest ones in the proof (but clearly not in this case!), but it is crucial to always check it.

Inductive Hypothesis (Assume that a domino piece falls) Assume that for some $n = k$, we have

$$1 + 2 + \dots + k = \frac{k(k+1)}{2}.$$

We will assume that the last equality is true. In general, once you assume the inductive hypothesis, you can assume that it holds whenever you need it. We show the explicit use of this hypothesis in the next step.

Inductive Step (Show that if a domino piece falls, then the next piece falls) We assumed that for some k , the following equality holds.

$$1 + 2 + \dots + k = \frac{k(k+1)}{2}.$$

We want to show that the domino piece following k falls. This piece is $k+1$. To show that it gets knocked off, we need to show that our equality holds if $n = k+1$. In other words, we need to show that

$$1 + 2 + \dots + k + (k+1) = \frac{(k+1)(k+2)}{2}.$$

Now look at our inductive hypothesis and at the equality we want to arrive to. Examine them and ask yourself: *What can we do to turn the inductive hypothesis into what we want?* In this case, if we look at the left sides of the equation, we see that there is an extra term in the sum we want to prove. The term is $k+1$, and the inductive hypothesis doesn't have it!

We need this $k+1$, so let's toss it into the inductive hypothesis. We do this in the line below. Since

$$1 + 2 + \dots + k = \frac{k(k+1)}{2},$$

we can add $k+1$ to both sides of the equation and get

$$1 + 2 + \dots + k + (k+1) = \frac{k(k+1)}{2} + (k+1).$$

Keeping in mind that we want the right side to read $\frac{(k+1)(k+2)}{2}$, we note that the last equation implies that

$$1 + 2 + \dots + k + (k+1) = \frac{k(k+1) + 2(k+1)}{2}.$$

But we can factor $k+1$ on the right side to get

$$1 + 2 + \dots + k + (k+1) = \frac{(k+2)(k+1)}{2},$$

which is exactly what we wanted to arrive to.

We showed that $n = 1$ satisfies the property and that if some k satisfies it, then $k+1$ satisfies it as well. This shows, by the induction principle, that our equation holds for every positive integer n , so we are done. \square

Now we will show another sum we can prove by induction. We will not annotate it as often as we did before, but it is important to note that the basic elements of an induction are still there, and have been clearly labeled.

Proposition 2. *Let n be a positive integer. Show that the quantity*

$$1 + 3 + 5 + 7 + \dots + (2n - 1)$$

is a perfect square.

Proof. The sum equals 1 when $n = 1$, so our base case is satisfied. However, to perform an induction, assuming that the sum equals any perfect square r^2 is not enough, since we would not be able to manipulate the expression

$$1 + 3 + 5 + 7 + \dots + (2n - 1) = k^2$$

into

$$1 + 3 + 5 + 7 + \dots + (2n - 1) + (2n + 1) = r^2$$

by simply adding $2n + 1$ to k^2 .

This means that we need to find something about *which* perfect square our sum equals. The only way to do this is to plug in some values of n and find a pattern. If $n = 2$, our sum is

$$1 + 3 = 4 = 2^2.$$

If we keep trying, we will see that the values of the sum for n equal to 3, 4 and 5 are, respectively 3^2 , 4^2 and 5^2 . Therefore, it is safe to *guess* that

$$1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2.$$

Let's prove this by induction.

We already took care of the basis step. Let's assume that for some n , it is true that

$$1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2.$$

This last statement is our inductive hypothesis. We must now show that

$$1 + 3 + 5 + 7 + \dots + (2n - 1) + (2n + 1) = (n + 1)^2.$$

To do this, we add $(2n + 1)$ to both sides of our inductive hypothesis to get

$$1 + 3 + 5 + 7 + \dots + (2n - 1) + (2n + 1) = n^2 + (2n + 1).$$

But we can arrange the right side of the last equation to get

$$1 + 3 + 5 + 7 + \dots + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2.$$

This is what we wanted to show, so our proof is complete. \square

Consider this other exercise.

Proposition 3. *For any integer n ,*

$$1^3 + 2^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2$$

We will not show this exercise, but you should try it at home. [Hint: Does the right side look familiar?]

Induction is useful for more than just proving sums. It is a very versatile tool, since we can construct millions of different statements depending on the natural numbers. Consider the following proposition.

Proposition 4. *Prove that $8^n - 1$ is divisible by 7 for any positive integer n .*

Proof. We can do this in two ways: One uses the properties of Z/nZ studied in class and one uses mathematical induction.

Proof by induction When $n = 1$, the statement asks us to show that 7 is divisible by 7. This statement is clearly true. We proceed by induction. Assume that for some n , 7 divides $8^n - 1$. Then there exists an integer m such that $8^n - 1 = 7m$. We need to show that there exists some integer r such that $8^{n+1} - 1 = 7r$.

We multiply the equation in the inductive hypothesis by 8 to turn the 8^n in it into 8^{n+1} in the equation we want to get. In other words, since $8^n - 1 = 7m$, we multiply by 8 to get $8^{n+1} - 8 = 56m$. Now we need to rearrange the expression until we reach the expression we are looking for. In this case, we note that $56m = 8^{n+1} - 8 = 8^{n+1} - 7 - 1$. This implies that $56m + 7 = 8^{n+1} - 1$, which implies that $7r = 8^{n+1} - 1$ for $r = 8m + 1$. This is what we wanted to show, so our proof is complete.

Proof with equivalence classes Showing that 7 divides $8^n - 1$ is equivalent to showing that the equivalence class of $8^n - 1$ in $Z/7Z$ is the equivalence class of 0. But now we note that, because of the properties of equivalence classes in this field,

$$[8^n - 1] = [8^n] - [1] = [8]^n - [1].$$

We conclude this proof by noting that $[8] = [1]$, so that

$$[8^n - 1] = [1]^n - [1] = [1^n] - [1] = [1] - [1] = [0].$$

This concludes our proof. □

Although equivalence classes gave us a much more concise argument in this case, the problem might be complicated enough to prevent us from finding a neat argument using equivalence classes. This would force us to settle with a less elegant yet equally valid proof by induction. This is the case in the following problem.

Proposition 5. *Prove that for any integer n , the number $11^{n+1} + 12^{2n-1}$ is divisible by 133.*

We will not prove this proposition since its technique is similar to the one used above. However, it is a very instructive exercise. [Hint: Multiply by 12 and rearrange].

We have seen that induction gives us a way to prove statements about the set of natural numbers or some of its subsets. But this is not an infallible method. Consider the following proposition.

Proposition 6. *Show that for any odd prime number p , either $p - 1$ or $p - 3$ is divisible by 4.*

Proof. We cannot proceed by induction (How do we know what the r th prime is?). However, the proof is quite simple. Any number t must be contained in some element of $Z/4Z$, so it must be of the form $4r, 4r + 1, 4r + 2$ or $4r + 3$ (Remember that forming equivalence classes in Z/nZ is the same thing as grouping numbers by their remainder when divided by n). But p may not be of the form $4r$ or $4r + 2$, since both of these numbers are even. Hence p is of the form $4r + 1$ or $4r + 3$. In either case, either $p - 1$ or $p - 3$ is divisible by 4. □

We hope that this short document provides a smooth introduction into what an *induction* is. Please do not hesitate to email any of your CA's if you have any questions about this document or about any aspect of the class. We're here to help! You will find some more challenging exercises at <http://www.maths.ox.ac.uk/prospective-students/undergraduate/practice-problems/pdf/allsheets2.pdf>.