SYSTEMS OF EQUATIONS

MATH 23A SECTION HANDOUT 3
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Solving linear systems of equations is one of the most immediate applications of linear algebra. This application is very realted to topics we will study later on in the semester so, throughout the handout, we will make reference to these topics. Although this handout assumes no more than what we have seen so far in class, we suggest that as the semester comes to an end, you review this handout and make sense out of the references to the things we haven't studied so far. We will discuss this over $\mathbb{R}^n$, but the results here are valid over any field. Furthermore, we focus on the *theoretical* aspects of solving systems. The examples and general idea for this handout are all found in the book used for Math 121, which is another good resource for us this semester.
Given an mxn matrix with real entries

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

we can define a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ by setting

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix}.$$

Note that the jth column of the resulting vector is obtained by multiplying each x_i by a_{ji} and adding the results together. In more technical terms we will see later in the semester, it is obtained by taking the dot product of the vector with erach product. We won't check that this is a linear map, but we will see this in class eventually.
Now consider a system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m, \end{aligned}$$

we can define the associated *coefficient matrix*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

Now, if

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},$$

we can rewrite this equation as

$$Ax = b.$$

By a *solution* to this equation, we mean a vector s for which if $x = s$, the equation above is valid. We want to know when a system of equations has a solutions and, whenever possible, to find an explicit formula that generates all of them. In other words, if solutions exist, we wish to *parametrize* them.
Our work begins studying *homogenous* systems of equations. A system of equations given in matrix form by $Ax = b$ as above is said to be *homogenous* if $b = 0$ and nonhomogenous otherwise. One of our main results is the following:

Theorem 1. *Let $Ax = 0$ be a homogenous system of equations with A as above, and let S be its set of solutions. Then S is a subspace of $\mathbb{R}^n$ of dimension $n - rank(A)$.*

Proof. We will give a sketch of a proof that will rely on results we will see later on in class. You should read it anyway.
Given a matrix A defined as above, we define its kernel to be the subspace consisting of all $x \in \mathbb{R}^n$ such that $Ax = 0$. Note that with this terminology, the solution set is simply the kernel of A. Let $null(A)$ denote the dimension of the kernel of A There is a theorem called the Rank-Nullity theorem which states that

$$rank(A) + null(A) = n,$$

and this implies that

$$null(A) = n - rank(A).$$

By definition, $null(A)$ is the dimension of the solution set of our system, so we are done. □

The last theorem helps us parametrize solutions of a homogenous system of equations. We know that the solution set is a vector space, so if we just find a basis for it, we will know how every single solution will look like!
However, our parametrization theorems go even further.

Theorem 2. *Let S be the solution set of a system of equations given in matrix form by $Ax = b$. Let T be the solution space of the corresponding homogenous equation $Ax = 0$. If s is a solution to $Ax = b$, then*

$$S = \{s + t : t \in T\}.$$

In other words, if we find a single solution to $Ax = b$ and the solution space for $Ax = 0$, we can parametrize all solutions for the system $Ax = b$. We will not prove this theorem, but we will give an example of how to apply it.
Consider the system

$$x_1 + 2x_2 + x_3 = 7x_1 - x_2 - x_3 = -4 \tag{1}$$

The coefficient matrix is

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}.$$

We can calculate the rank of A to obtain $rank(A) = 2$.
Let's focus on the homogenous system $Ax = 0$. By the results above, its solution set is a vector space of dimension $3 - 2 = 1$. Since

$$x = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

is a solution to this homogenous system, it follows that the solution space is

$$span \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

In other words, any vector in the solution set is of the form

$$t \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

for $t \in \mathbb{R}^n$.
We are done solving the homogenous system. To solve the nonhomogenous system, note that a particular solution to

$$Ax = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$

is given by

$$x = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}.$$

Hence, by the preceding results, the solution set to our original system is given by

$$S = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

That's it, we're done!
But *when* can we find a particular solution to the system $Ax = b$ so we can parametrize all solutions? The answer is in the following theorem.

Theorem 3. *A system of equations $Ax = b$ has a solution if and only if*

$$rank \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix} = rank \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

We will see a fairly easy proof of this when we have a better understanding of linear transformations. For now, just note that in the example we just solved, we know that a particular solution to the system exists because

$$rank \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix} = rank \begin{pmatrix} 1 & 2 & 1 & 7 \\ 1 & -1 & -1 & -4 \end{pmatrix} = 2.$$