

WHAT IS A PROOF?

MATH 23A SECTION HANDOUT 1
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A proof is a sequence of true statements that give a complete, irrefutable justification of a statement. In other words, a proof tells us why something works. Proofs are at the heart of mathematics, and provide the foundations on which every new concept is built.

Any mathematical theory (arithmetic, calculus, linear algebra...) begins with a set of definitions and axioms. These are related with each other by some results (lemmas, propositions, theorems...), and each of these results is backed up by an explanation that explains why it is true. This explanation is a proof. Consider the following statement.

Theorem 1. *If x is a real number, then $x^2 + 1 \geq 2x$.*

Why is this true? This seems true since plugging in values of x confirms that it works. We could try hundreds of thousands of values, but this would still not be enough to guarantee that the statement will *never* fail. We need something more *general*. The following argument shows us this more general justification.

Proof. We know that the square of any real number is non-negative. In particular, this means that for any x , we have

$$(x - 1)^2 \geq 0.$$

Expanding the square term, we obtain

$$x^2 - 2x + 1 \geq 0.$$

Rearranging, this implies

$$x^2 + 1 \geq 2x.$$

This is exactly what we wanted to show. □

This proof is a valid justification of Theorem 1. Every step is clearly implied by the step above it and it ultimately uses no more than the conditions (usually called *hypothesis*) given by the statement and our general knowledge about its context (in this case, the real numbers). The aim of a proof is to achieve what these two things: to have every step imply the next and to not assume any information that makes us lose generality.

For example, if we had ever assumed in the last proof that x was positive and completed the proof for this case, we would need to verify the statement when x is non-positive. Otherwise we would not be able to assure that the statement is *always* true.

How did we know that we had to start by writing $(x - 1)^2 \geq 0$ Although this choice seems very arbitrary, reading the proof from its end to its beginning will clarify how we knew where to start. When beginning a proof, we should always know exactly *what* we want to show. Once this is clear, we should think about

emphow we are going to do it. It is a game, and should not be considered an arduous task. With practice, proving certain things will become very natural.

There is no general method to prove any statement (Math would be very boring if there were!). However, it is not uncommon to build on previous proofs to come up with new ones. Consider the following statement.

Theorem 2. *For any real numbers x and y , $x^2 + y^2 + 5 \geq 2x + 4y$.*

Proof. By Theorem 1, $x^2 + 1 \geq 2x$. In addition, we know that for any real number y ,

$$(y - 2)^2 \geq 0.$$

This implies that

$$y^2 + 4 \geq 4y.$$

If we add this last expression to the result in Theorem 1, we obtain the desired inequality. \square

In this ways, proofs build on themselves, and increasingly more complicated proofs arise from simpler ones. As proofs get more and more complicated, rephrasing our original statement might provide some insight as to *how* to prove it. The next sections illustrate some ways of rephrasing an argument.

1. CONTRADICTIONS

Proving a statement by contradiction means showing that if its conclusion doesn't hold, then something impossible would happen. We will study Theorem 2 in this context. Its conclusion reads $x^2 + y^2 + 5 \geq 2x + 4y$. A proof by contradiction of Theorem 2 would read in this way.

Proof. Assume that the inequality does not hold in the direction the statement shows. In other words, assume that

$$x^2 + y^2 + 5 < 2x + 4y.$$

Rearranging, this implies that

$$x^2 - 2x + 1 + y^2 - 4y + 4 < 0.$$

Grouping the terms in the last inequality gives

$$(x - 1)^2 + (y - 2)^2 < 0.$$

However, this implies that the sum of the squares of two real numbers is *strictly* less than zero. This is impossible, so we have reached a contradiction. This means that our assumption is false, and so the original statement is true. \square

Be careful when negating statements. For example, the phrase “for every a , property P holds” negates into “there exists some a such that P doesn't apply to it.”

2. CONTRAPOSITIVE

We will use an example to explain what the contrapositive of a statement is.

Theorem 3. *If $x > 0$, then x^3 is nonzero.*

We will not prove this. Instead, we will simply show its contrapositive.

Theorem 4. *If x^3 is zero, then $x \leq 0$.*

This example illustrates what proving a statement by proving its contrapositive is all about: Proving that some conditions C imply a conclusion Q is *equivalent* to proving that the negation of Q implies the negation of the set of conditions C .

While contradictions try to find any false conclusion we can arrive to from the negation of Q , contrapositives explicitly try to show that if Q doesn't happen, then C doesn't hold at all. For example, we show the contrapositive of Theorem 2 in the following theorem.

Theorem 5. *If $x^2 + y^2 + 5 < 2x + 4y$, at least one of x and y is not real.*

3. TIPS AND STRATEGIES

Everyone thinks differently, and there are as many approaches to proofs as there are personalities. However, there are some useful things we can do before or while proving a statement.

- Read carefully and make sure we know *WHAT* we are asked to do.
- Think of what results would help imply the statement's conclusion.
- Make sure we use all the conditions the statement gives us.
- Verify (and show) that the conditions of a theorem are satisfied before using it.
- Write a concise but thorough explanation of our thoughts that clearly shows every step.
- If an idea doesn't work, we should avoid falling into a vicious cycle (using the same idea over and over again without noticing instead of trying new ones).
- You should stop writing when you feel that anyone who reads your work would understand each of your ideas and understands why each of the steps you took is valid. Don't be afraid to write a bit more than you expect: Write as much as you need to convince yourself that no one would be able to claim that you are wrong.
- BE CREATIVE and HAVE FUN!!!!